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CLOSURE UNDER THE MAJORIZATION RELATION AND THE DISTINGUISHING CHROMATIC NUMBER OF CIRCULANT GRAPHS

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BY

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ABSTRACT

This dissertation addresses two distinct problems in graph theory and in each case advances results for invariants of graphs. The first problem investigates the arrangement of the degree sequences of various classes of graphs in the dominance order. In the second problem we consider a family of graphs, best described as a generalization of cycle graphs, and investigate the values of the distinguishing chromatic number for the complements of these graphs. An improvement of a known bound for this number is also given for this particular class of graphs.

Chapter 1 introduces fundamental definitions, concepts, notations and known results that are used throughout the rest of the thesis.

In Chapter 2, we take a look at how degree sequences of various classes of graphs are ordered by the majorization relation. This ordering gives rise to a poset known as the dominance order within which the degree sequences of threshold and split graphs were shown by Hammer et al. and Merris to display an orderly arrangement. To give context to these examples and better understand how degree sequences of classes of graphs situate themselves in the dominance order, we define what it means for a collection of graphs $\mathcal{F}$ to be dominance monotone. Furthermore, we characterize the dominance monotone sets $\mathcal{F}$ for "small" families $\mathcal{F}$, and as a result two new classes of graphs whose degree sequences form an upward-closed set in the dominance order are discovered and identified.

Chapter 3 is dedicated to the study of the distinguishing chromatic number for the complements of circulant graphs $C_n(1,k)$, a family of graphs formed by adding a set of chords to a cycle. In general, we use our knowlege of the graph structure, and known and proven results such as the automorphism group of the graph to come up with constructions that determine the distinguishing chromatic number of their complements. These results, together with previous results on

the distiguishing chromatic number for the circulant graphs $C_n(1, k)$ from a joint work done with Barrus and Lantz, provide an improvement for upper bounds on the sum and product of the distinguishing chromatic numbers of these graphs and their complements produced by Collins and Trenk.

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This dissertation, formally characterized as a personal achievement, wouldn't be in fact possible without the support, assistance and advice of a great deal of wonderful Souls; some of them I would like to dedicate a couple sentences to.

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Someone who needs no introduction (for his body of work speaks for itself) that I would like to also thank is Dr. Mustafa Kulenović. From day one, he made it clear to me that his door is always open for whenever I want to talk math and life in general, and whenever I need help with job recommendations. As his former student, Dr. Elliott Bertrand, describes him: "He is a true Renaissance man". I was fortunate to have access to and be taught in several occasions by him.

The names of the next two Souls that I am about to address are Dr. Christopher Hunter and Assistant Dean Earl N. Smith III. Dr. Christopher Hunter is a civil engineer and Assistant Dean Earl N. Smith III is in charge of the student affairs in the College of Arts and Sciences. They are both African American faculty members at the University of Rhode Island. I must say that meeting and having access to these two gentlemen was truly a blessing; for as a Haitian immigrant and a Black man there exist conversations that I find more comfortable having with them. They both have always been inspiring and motivating spirits to me. On top of that, Dr. Christopher Hunter serves on my dissertation committee. Thus, kudos to both of them. To them, I am very grateful.

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Instrumental to my success as a graduate student and a PhD candidate has been the warm hospitality of the Department of Mathematics at the University of Rhode Island. The kindness and goodness of its residents cannot be overstated. The tone is set by the Chair of the department, Dr. James Baglama, and the administrator Mrs. Deborah Beagan, who arguably are two of the nicest people that have ever walked the face of this earth. I would also like to thank all the people I have learnt a lot from: Dr. John Velling (my undergraduate mentor), Dr. Jeff Suzuki, Dr. Johnny Guzmán, Dr. William Massey, Dr. Anthony Clement, Dr. Barbara Kaskosz, Dr. Marc Comerford, Dr. Tom Bella, Dr. Araceli Bonifant, Dr. Lubos Thoma, Dr. Li Wu, Dr. Tom Sharland, Dr. Bill Kinnersley, Dr. Orlando Merino, Laura Barnes, Robin Schipritt, Dr. Erin Denette, and all my fellow graduate students. I would like to specially thank our wonderful Information Technologist, Heath Loder, who, on so many occasions, has literally saved my life.

To the University of Rhode Island and the Ocean State taxpayers, I would like you to know that I am forever indebted to you for this opportunity.

Last but not least, I would like to thank my family, Mom and Dad, who gave up all they had in order for us to have the best elementary and secondary education we could possibly have in Haiti. I would like to dedicate this work especially to Mom, who got diagnosed with stage 4 cancer six months before I started my graduate studies. Her words when I suggested to her that I would pass on the opportunity to go to grad school, so I can take care of her, were ”Pa pemet ou” (which stands for “I dare you” in Haitian creole). Of course, I save the best for last: special thanks to my rock, my champion, my son, Mr. Prinz S. Guillaume. No one has suffered more during my time away in grad school. Father and son’s activities have been reduced to a bare minimum and yet this Dude remains my biggest supporter throughout this process. I simply cannot put into words how much your understanding and support meant to me. I hope to live long enough, so I can make it up to you. I love you, Son.

DEDICATION

To my dear Mama who has been a source of inspiration to me through this journey of becoming a mathematician: the bright smile you have managed to maintain in the midst of your battle with cancer reminds me every day how lucky I am to be healthy and be able to pursue my passion.

PREFACE

This thesis has been prepared in manuscript form. The main content of the thesis includes two research papers, Manuscripts 1 and 2. Manuscript 1, covered in Chapter 2, was submitted for publication om May 22nd, 2019, and Manuscript 2, covered in Chapter 3, will be submitted soon.

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CHAPTER 1

Introduction

1.1 Graphs: Definition

Graph theory is the mathematical analysis of networks. It is used in any application where configurations of nodes and connections occur. Many of these applications which are laid out in [4, 5] include, but are not limited to, electrical circuits, roadways, organic molecules, or sociological relationships. Such configurations are modeled by combinatorial structures called graphs. In this chapter, we introduce some fundamental concepts, basic notions and definitions relevant to the study at hand.

A graph $G = (V, E)$ is a mathematical structure which consists of two finite sets V, the set of vertices, and E the set of edges. The number of vertices in G is denoted $|V(G)|$, which is also the order n of G, and the number of edges is denoted $|E(G)|$. Two vertices with an edge between them are said to be *adjacent* or *neighbors* and the number of edges incident to a vertex v is called the *degree of* v, denoted $d(v)$. A vertex that is adjacent to all other vertices in a graph is said to be *dominating*. In certain applications of graph theory and in some theoretical contexts, there exist frequent instances where *loops* (an edge joining a vertex to itself), directed edges (edges together with a direction), or multiple edges between a pair of vertices arise. However, in this work, we restrict ourselves to *simple* graphs; that is, we do not allow loops, directed edge and multiple edges. Figure 1 shows two simple graphs G, H. In G, the degrees of vertices u and v are 2, and 1 respectively.

The list $d = (d_1, \cdots, d_n)$ of the vertex degrees of a graph G is called the *degree sequence* of G and the terms of the list are often written in nonincreasing order for convenience. For example, in Figure 1, $d_G = (2, 2, 1, 1)$ and $d_H = (2, 2, 2)$. At times, particularly when degree sequences appear in pairs, we will write specific degree sequences with small terms without parentheses or commas, as in $d = d_1 \cdots d_n$. For multiple identical terms within a degree sequence we may use exponents to indicate multiplicities.

Any graph having a list d as its degree sequence is called a *realization* of d. Note that a degree sequence may have two or more *nonisomorphic* graphic representations or realizations, which are graphs that are structurally distinct. As an example, Figure 2 displays the two nonisomorphic graphs that realize the degree sequence $d = 22211$. We are primarily interested in properties of graphs that do not change when vertices are relabeled; in other words properties that are invariants under isomorphism. Recall that two graphs G_1 and G_2 are said to be isomorphic if there exists a structure-preserving vertex bijection between them; that is if there exists a matching between their vertices so that two vertices are adjacent in G_1 if and only if corresponding vertices are adjacent in G_2. An isomorphism from a graph G to itself is called an *automorphism* and the set all automorphisms of G is known to form a group denoted $\mathrm{Aut}(G)$ under composition as a binary operation. In Chapter 3, we take advantage of our familiarity with this set to achieve one of our stated goals, which is to use the least number of labels possible to destroy all the nontrivial automorphisms of G and thereby render all vertices distinguishable or fixed.

One elementary but huge result in graph theory is that the sum of the vertex degrees in a graph is even (see [1]). This result is better known as the *handshake lemma* and is referred to in Chapter 2 of this work. It simply follows from the fact that every edge contributes the value of 2 to this sum.

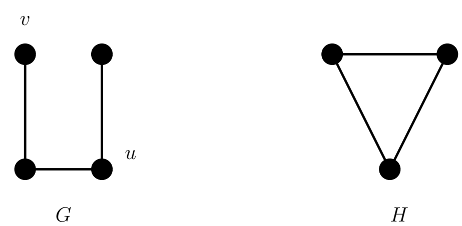

Figure 1. $d_G = 2211$ and $d_H = 222$; $d(u) = 2$ and $d(v) = 1$.

1.2 Graph Complements, Induced and Forbidden Subgraphs

Also of significance to us in both studies are the notions of *graph complements*, *induced subgraphs* and *forbidden subgraphs* of a graph. The complement $\overline{G}$ of a graph G is the graph on the same vertex set, such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Based on this definition, we see that the complement of H (the triangle graph in Figure 1) is the graph that consists of three isolated vertices known as $3K_1$ and the graph G in the same figure is self-complementary.

An induced subgraph of a graph is a subgraph obtained as a result of vertex deletion. For example, if we delete one vertex of degree 1 in H' in Figure 2, we obtain G in Figure 1. Thus G is an induced subgraph of graph H'. Also, in chapter 2 we refer to the term, *induced maximum matching*, to indicate the maximum size of an independent(mutually non-adjacent) set of edges that can be obtained from a graph via vertex deletion. On the other hand, a graph G may not allow certain graph(s) as induced subgraphs. If a graph G does not contain a graph F as an induced subgraph, we say that G is F-free. As an illustration, we can say that the graph G in Figure 1 is H-free, where H is the triangle graph in the same figure. Moreover if no realizations of a degree sequence d contains F as an induced subgraph, we say that d is forcibly F-free. Thus, the degree sequence $d = 2211$ of G in Figure 1 is forcibly H-free. Similarly, given $\mathcal{F}$ a set of graphs, we say that a graph G is $\mathcal{F}$-free if G does not contain any graph of $\mathcal{F}$ as an induced subgraph.

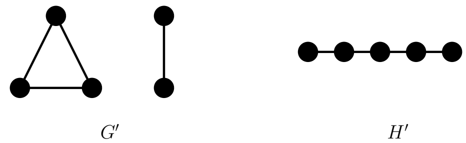

Figure 2. G' is the union of a triangle(K_3) and a path on 2 vertices(K_2); H' is a path on 5 vertices or P_5.

1.3 Classes of Graphs

In graph theory circles, some classes of graphs enjoy special names such as the class of complete graphs, K_n, in which every pair of its n vertices is adjacent. Our graph H in Figure 1 is an example of complete graph and is called K_3. The set of vertices of a complete graph $V(K_n)$ form what is called a *clique*, which is a maximal subset of mutually adjacent vertices in a graph. A *path* P_n is a trail on n distinct vertices. G in Figure 1 and H' in Figure 2 are examples of paths. The class of *cycle graphs* C_n is a nontrivial closed path like H in Figure 1 which is also a C_3. The *trees* make up the class of connected acyclic graphs. Two other classes of graphs that draw a lot of attention in chapter 2 are the *split* and *threshold* graphs: [2] describes the split graph as a graph whose vertex set can be partitioned as the disjoint union of an independent set and a clique; in [3] a threshold graph is defined as a graph that can be constructed starting from a single vertex and sequentially adding a finite number of either dominating or isolating vertices. Last but not least, we have the class of *bipartite* graphs, which are the graphs whose vertex sets can be partitioned into two subsets U and W, such that each edge of the graph has one endpoint in U and one endpoint in W.

In this work, we use $+$ between two graphs to symbolize the union of these graphs and $\vee$ to symbolize the join which connects each vertex of one graph to all the vertices of the other. In Figure 2, G' can be refered as $K_3 + P_2$ and in Figure 1, H can be also named $K_1 \vee K_2$. We often use aG to denote the disjoint union of a copies of G. This is the end of our chapter on the fundamental concepts and basic notions in graph theory. All other useful concepts are described in the chapter where they are used.

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CHAPTER 2

Upward-Closed Hereditary Families in the Dominance Order

Michael D. Barrus and Jean Guillaume

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Abstract

The majorization relation orders the degree sequences of simple graphs into posets called dominance orders. As shown by Hammer et al. and Merris, the degree sequences of threshold and split graphs form upward-closed sets within the dominance orders they belong to, i.e., any degree sequence majorizing a split or threshold sequence must itself be split or threshold, respectively. Motivated by the fact that threshold graphs and split graphs have characterizations in terms of forbidden induced subgraphs, we define a class $\mathcal{F}$ of graphs to be *dominance monotone* if whenever no realization of e contains an element $\mathcal{F}$ as an induced subgraph, and d majorizes e, then no realization of d induces an element of $\mathcal{F}$. We present conditions necessary for a set of graphs to be dominance monotone, and we identify the dominance monotone sets of order at most 3.

2.1 Introduction

In this paper, we study the interactions of two aspects of graph degree sequences, namely their relationships under the majorization order, and the induced subgraphs that their realizations may or must not contain.

When degree sequences with a common sum are ordered via majorization, interesting observations are possible. Here we assume that $d = (d_1, \ldots, d_n)$ and $e = (e_1, \ldots, e_p)$ are lists of positive integers with their terms in nonincreasing order, and we say that d *majorizes* e, denoted $d \succeq e$, if

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{p} e_i \quad \text{and} \quad \sum_{i=1}^{k} e_i \leq \sum_{i=1}^{k} d_i \text{ for } 1 \leq k \leq \min\{p, n\}.$$

Applying the relation $\succeq$ to all partitions of a fixed positive integer yields a poset. As observed by Ruch [8] and others, all *graphic* partitions (i.e., degree sequences of simple graphs) among these partitions form an ideal, or downward-closed set, meaning that if d is a degree sequence and $d \succeq e$, then e is a degree sequence as well. As an illustration, refer to Figure 3.

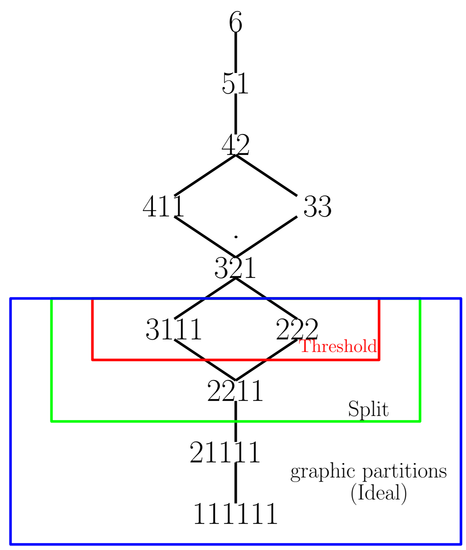

Figure 3. Partitions of positive integers ordered by the majorization relation

If we restrict our attention to the portion of the majorization poset containing just the graphic partitions, we obtain the *dominance order* on degree sequences having a common sum. The degree sequences near the top of the dominance order belong to interesting graph classes. The maximal degree sequences in the dominance order are known as the *threshold sequences*, and their realizations, the *threshold graphs*, have been shown to have several remarkable properties (see the monograph [4] for a survey). Merris [6] showed that the more general class of *split graphs*, those whose vertex sets can be partitioned into a clique and an independent set, have degree sequences that are upward-closed in the dominance order, meaning that if e is the degree sequence of some split graph and d is any degree sequence majorizing e, then every realization of d is a split graph as well.

In addition to their degree sequence characterizations, the classes of threshold graphs and of split graphs both have characterizations in terms of induced subgraphs. Chvátal and Hammer [2] showed that threshold graphs are precisely those graphs that are $\{2K_2, C_4, P_4\}$-*free*, meaning that these graphs have no induced subgraph isomorphic to any of $2K_2$, C_4, or P_4. Földes and Hammer [3] likewise showed that the split graphs are the $\{2K_2, C_4, C_5\}$-free graphs.

Recently [1], the *weakly threshold graphs* were introduced by the first author as those graphs for which the degree sequences satisfied a relaxation of a degree sequence characterization of threshold graphs. Weakly threshold graphs form a subclass of the split graphs, and like the split and threshold graphs, they have a forbidden subgraph characterization and the property that any degree sequence majorizing the degree sequence of a weakly threshold graph is itself the degree sequence of a weakly threshold graph.

In light of these examples, it appears that we may better understand one facet of the dominance order by considering hereditary graph classes like the threshold, split, and weakly threshold graphs whose degree sequences form upward-closed sets in the dominance order. To do this, we will focus on the corresponding sets of forbidden induced subgraphs. We define a set $\mathcal{F}$ of graphs to be *dominance monotone* if the following property is true:

If d and e are degree sequences such that $d \succeq e$ and every realization of e is $\mathcal{F}$-free, then every realization of d is $\mathcal{F}$-free as well.

In other words, $\mathcal{F}$ is dominance monotone if the forcibly $\mathcal{F}$-free-graphic sequences form an upward-closed set in each dominance order (precise definitions will be given in the following section).

In this paper we initiate the study of dominance monotone sets, establishing necessary conditions and determining all dominance monotone sets of size at most 3. In Section 2, we recall preliminary notation, definitions, and results on degree sequences, majorization, and forbidden subgraphs. In Section 3 we determine necessary conditions for graphs in dominance monotone sets and use these conditions to determine the dominance monotone sets of order 1. In Sections 4 and 5 we characterize the dominance monotone sets $\mathcal{F}$ for which $|\mathcal{F}| = 2$ and $|\mathcal{F}| = 3$, respectively, including the first known dominance monotone examples $\mathcal{F}$ for which the $\mathcal{F}$-free graphs are not a subclass of the split graphs. In Section 6 we present a few concluding remarks and questions.

2.2 Preliminaries

In this section, we recall basic terminology and notions for degree sequences and related concepts.

All graphs considered here are finite and simple. We denote the vertex set and edge set of a graph G, respectively, by $V(G)$ and $E(G)$, and we define $n(G) = |V(G)|$. We use $\overline{G}$ to denote the complement of G.

For any $v \in V(G)$, we use $d_G(v)$ to denote the degree of v in G, and we write the degree sequence of G as a list $d_G = (d_1, d_2, \ldots, d_n)$ having terms in nonincreasing order. At times, particularly when degree sequences appear in pairs, we will write specific degree sequences with small terms without parentheses or commas, as in $d = d_1 d_2 \cdots d_n$. For multiple identical terms within a degree sequence we may use exponents to indicate multiplicities. We set $\Delta(G) = d_1$ and $\delta(G) = d_n$.

Any graph having such a list d as its degree sequence is called a *realization* of d. (Graphs in this paper are unlabeled, meaning that we are not careful to distinguish between isomorphic realizations of a degree sequence).

Turning now to majorization, we use $\mathcal{D}_{2m}$ to denote the dominance order on graphic partitions of $2m$, where m is an integer; it is an elementary result that the sum of the terms in any degree sequence is an even number. We will assume that all terms in elements of $\mathcal{D}_{2m}$ are positive; though of course some graphs do contain isolated vertices, we emphasize that realizations of elements in $\mathcal{D}_{2m}$ are assumed not to.

We may illustrate degree sequences in $\mathcal{D}_{2m}$ and their relationships under majorization using a geometric description known as a *Ferrers diagram.* For

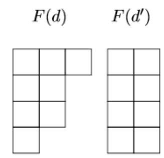

Figure 4. Ferrers diagrams of $d = 3221$ and $d' = 2222$.

$d = (d_1, \ldots, d_n) \in \mathcal{D}_{2m}$, define the Ferrers diagram $F(d)$ as a left-justified array made up of $2m$ boxes arranged into rows, with the ith row of $F(d)$ consisting of d_i boxes for $i \in \{1, \ldots, n\}$. As an illustration, Figure 4 displays the Ferrers diagrams of $d = 3221$ and $d' = 2222$.

A fundamental result on partitions known as Muirhead's Lemma [7] can be recast as the following statement involving Ferrers diagrams: two degree sequences $d, d' \in \mathcal{D}_{2m}$ satisfy $d \succeq d'$ if and only if $F(d')$ can be obtained from $F(d)$ by moving one or more boxes down to lower rows (even if this process gives rise to new rows) while ensuring that the numbers of boxes in the rows remain in nonincreasing order. In Figure 4, moving a box from the first row to the fourth row of $F(d)$ yields $F(d')$; hence, $3221 \succeq 2222$.

We say that a class of elements in a dominance order $\mathcal{D}_{2m}$ is *upward-closed* if whenever d and e are elements of $\mathcal{D}_{2m}$ such that e belongs to the class and $d \succeq e$, it follows that d belongs to the class as well. For an upward-closed class of degree sequences, Muirhead's Lemma implies that moving any box in the Ferrers diagram of one of these degree sequences to an earlier row produces the Ferrers diagram either of another degree sequence in the class or of a non-graphic partition.

When we consider realizations of degree sequences, it is important to note that a single degree sequence may have multiple nonisomorphic realizations. For this reason, for any graph-theoretic property $\mathcal{P}$ invariant under isomorphism, we say that a degree sequence d is *potentially $\mathcal{P}$-graphic*, or *potentially $\mathcal{P}$*, if at least one of the realizations of d has property $\mathcal{P}$. If every realization of d has property $\mathcal{P}$, we say that d is *forcibly $\mathcal{P}$-graphic*, or *forcibly $\mathcal{P}$*. Thus if $\mathcal{F}$ is a collection of graphs, we say that a degree sequence d is forcibly $\mathcal{F}$-free if no realization of d contains any element of $\mathcal{F}$ as an induced subgraph.

2.3 Necessary conditions and dominance monotone singletons

We work now towards characterizing dominance monotone sets. Recall that a collection $\mathcal{F}$ of graphs is dominance monotone if the class of forcibly $\mathcal{F}$-free sequences is upward-closed in each dominance order $\mathcal{D}_{2m}$.

Since our objective is to identify the dominance monotone sets, we say that a pair (d, e) of degree sequences is a *counterexample pair for $\mathcal{F}$* if $d \succeq e$ and e is forcibly $\mathcal{F}$-free, but d is not, i.e., d has a realization containing an element of $\mathcal{F}$ as an induced subgraph. There is a counterexample pair for $\mathcal{F}$ if and only if $\mathcal{F}$ is not dominance monotone.

For example, the set $\mathcal{F} = \{2K_2, C_4\}$ is not dominance monotone, since the dominance order $\mathcal{D}_{10}$ yields the counterexample pair $(32221, 2^5)$, in which 2^5 has the chordless 5-cycle (which contains neither $2K_2$ nor C_4 as induced subgraphs) as its only realization, and 32221 has as one of its realizations a chordless 4-cycle with an attached pendant vertex. Since the set $\{2K_2, C_4\}$ is the set of induced subgraphs forbidden for the *pseudo-split graphs*, which further have a degree sequence characterization (see [5]), we see that not every hereditary family with a degree sequence characterization forbids a dominance monotone set; more importantly, we also see that dominance monotone sets like $\{2K_2, C_4, C_5\}$ and $\{2K_2, C_4, P_4\}$ may contain non-dominance monotone subsets.

Our first result deals with complements. We use $\overline{G}$ to denote the complement of a graph G, and, given a collection $\mathcal{F}$ of graphs, we define $\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$.

Theorem 2.3.1 *If $\mathcal{F}$ is dominance monotone and no graph in $\mathcal{F}$ has a dominating* *vertex, then* $\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$ *is dominance monotone as well.*

Proof. Assume that $\mathcal{F}$ is dominance monotone and contains no graph with a dominating vertex. Suppose that $e = (e_1, \ldots, e_p)$ is forcibly $\overline{\mathcal{F}}$-free and $d \succeq e$, where $d = (d_1, \ldots, d_n)$.

Suppose first that $e_1 < p-1$. Form $\overline{e} = (p-1-e_p, \ldots, p-1-e_1)$, the degree sequence of the complement of any realization of e, noting that every term of $\overline{e}$ is positive. Muirhead's Lemma implies that $n \leq p$. Now form $\overline{d} = ((p-1)^{p-n}, p-1-d_n, \ldots, p-1-d_1)$; this is the degree sequence of the graph formed by adding $p-n$ isolated vertices to a realization of d and then taking the complement of the resulting graph.

Note that $\overline{e}$ is forcibly $\mathcal{F}$-free. Since every term in $\overline{e}$ or $\overline{d}$ is positive, and $\overline{e}$ and $\overline{d}$ are both partitions of $p(p-1) - \sum e_i$, they belong to the same dominance order; furthermore, for each $k \in \{1, \ldots, p\}$,

$$\sum_{i=1}^{k} \overline{e_i} = k(p-1) - \sum_{i=p+1-k}^{p} e_i = k(p-1) - \left(p(p-1) - \sum_{i=1}^{p-k} e_i\right) = \sum_{i=1}^{p-k} e_i - (p-k)(p-1)$$
$$\leq \sum_{i=1}^{p-k} d_i - (p-k)(p-1) = \sum_{i=1}^{k} \overline{d_i}.$$

Hence $\overline{d}$ majorizes $\overline{e}$. Since $\mathcal{F}$ is dominance monotone, $\overline{d}$ is forcibly $\mathcal{F}$-free, and the complement of any of its realizations is forcibly $\overline{\mathcal{F}}$-free. It follows that d is forcibly $\overline{\mathcal{F}}$-free, as claimed.

Suppose now that $e_1 = p-1$. Form $\overline{e}' = (p, p-e_p, \ldots, p-e_1)$ and $\overline{d}' = (p^{p-n+1}, p-d_n, \ldots, p-d_1)$; these are precisely the sequences $\overline{e}$ and $\overline{d}$ from the previous paragraph, but with each term increased by one and an extra term of p inserted at the beginning. Each term of $\overline{e}'$ and of $\overline{d}'$ is positive, and similar arguments to those above show that $\overline{d}' \succeq \overline{e}'$. If $\overline{e}'$ is forcibly $\mathcal{F}$-free, then $\overline{d}'$ will be forcibly $\mathcal{F}$-free and hence d will be forcibly $\overline{\mathcal{F}}$-free, as desired, since any realization of d is an induced subgraph of some realization of the complement of $\overline{d}'$. It suffices, then, to note that any realization of $\overline{e}'$ is obtained by adding a dominating vertex to the complement of a realization of e. Since no graph in $\mathcal{F}$ has a dominating vertex, if $\overline{e}'$ induces an element of $\mathcal{F}$, the vertices of this induced subgraph must include only vertices not of degree p in $\overline{e}'$. However, the subgraph induced on such vertices is the complement of an $\overline{\mathcal{F}}$-free graph, a contradiction. □

Theorem 2.3.2 *In every dominance monotone set, the graph with the lowest number of edges has maximum degree less than or equal to 1.*

Proof. Let $\mathcal{F}$ be a dominance monotone set. If all graphs in $\mathcal{F}$ with the lowest number of edges have maximum degree greater than 1, then for such a graph F, the pair $(d(F), 1^{2|E(F)|})$ is a counterexample pair, since no element of $\mathcal{F}$ is induced in a realization of $1^{2|E(F)|}$, which is a contradiction. □

Corollary 2.3.3 *If $\mathcal{F}$ is a dominance monotone set, then $\mathcal{F}$ contains either a graph with a dominating vertex or a $(|V(F)|-2)$-regular graph F; in the latter case F has an even number of vertices.*

Proof. Let $\mathcal{F}$ be a dominance monotone set in which no graph has a dominating vertex. By Theorem 2.3.1, $\overline{\mathcal{F}}$ is also a dominance monotone set. By Theorem 2.3.2, there exists a graph in $\overline{\mathcal{F}}$ with maximum degree at most 1. The complement of this graph is in $\mathcal{F}$; call it F. Thus, any vertex degree d of F satisfies $d \geq |V(F)|-1-1$. Since F has no dominating vertex, we also have $d \leq \Delta(F) \leq |V(F)|-2$; thus F is $(|V(F)|-2)$-regular. Since the sum of degrees in a graph is always even, $|V(F)|$ must be even. □

Recall from Section 1 that the threshold sequences are the maximal elements of a dominance order, and that their realizations are precisely the $\{2K_2, C_4, P_4\}$-free graphs.

Proposition 2.3.4 *If a collection $\mathcal{F}$ of graphs contains an induced subgraph from each $2K_2$, C_4, and P_4, then $\mathcal{F}$ is dominance monotone.*

Proof. Assume that each of $2K_2, C_4, P_4$ has an induced subgraph belonging to $\mathcal{F}$. Every forcibly $\mathcal{F}$-free sequence is then a threshold sequence and is not majorized by any other degree sequence. Thus no counterexample pair exists for $\mathcal{F}$, and $\mathcal{F}$ is dominance monotone. □

We can now characterize the dominance monotone sets with size 1.

Theorem 2.3.5 *The dominance monotone sets of cardinality 1 are $\{K_1\}, \{2K_1\}$, and $\{K_2\}$.*

Proof. By Proposition 2.3.4, $\{K_1\}, \{K_2\}, \{2K_1\}$ are all dominance monotone sets. Let $\mathcal{F} = \{F\}$ be a dominance monotone set. By Theorem 2.3.2, $\Delta(F) \leq 1$. If F has a dominating vertex then F equals K_1 or K_2; otherwise, by Corollary 2.3.3, F is $(|V(F)| - 2)$-regular. This implies that $|V(F)| - 2 \leq 1$, and since F has an even number of vertices, F must be $2K_1$. □

2.4 Dominance monotone pairs

Because graphs with maximum degree at most 1 are necessary elements in dominance monotone sets, by Theorem 2.3.2, we begin this section by establishing a result related to them.

Lemma 2.4.1 *Let $a, b \geq 0$ with $b \geq 3$ if $a = 0$ and $b \geq 1$ if $a = 1$. If $\mathcal{F}$ is a dominance monotone set containing $aK_2 + bK_1$, then $\mathcal{F}$ contains an induced subgraph of a disjoint union of cycles having at most $3a + 2b - 1$ vertices.*

Proof. Assume that $a, b \geq 0$ with $b \geq 3$ if $a = 0$ and $b \geq 1$ if $a = 1$. Assume also that $\mathcal{F}$ is a dominance monotone set containing $aK_2 + bK_1$. Consider the degree sequences $d = 3^1 2^{3a+2b-3} 1^1$ and $e = 2^{3a+2b-1}$ and note that d majorizes e. We claim that the degree sequence d is not forcibly $\mathcal{F}$-free. If $a = 0$, one realization is the graph obtained by adding the edge v_1v_{2b-2} in a path $v_1v_2\cdots v_{2b-1}$; deleting v_{2i} for all $1 \leq i \leq b-1$ leaves bK_1 as an induce subgraph. If $a \geq 1$, one realization is the graph obtained by adding the edge v_1v_3 to the path $v_1v_2\cdots v_{3a+2b-1}$; deleting v_{3i} for all $1 \leq i \leq a$ and v_{3a+2j} for $1 \leq j \leq b-1$ (when these vertices exist) leaves $aK_2 + bK_1$ as an induced subgraph.

Every realization of the degree sequence e is a disjoint union of cycles. Note that if $aK_2 + bK_1$ were induced in a disjoint union of cycles on $3a + 2b - 1$ vertices, we could arrive at such a subgraph by deleting $a+b-1$ vertices; howevever, deleting $a + b - 1$ vertices from a disjoint union of cycles leaves an induced subgraph with at most $a + b - 1$ components.

Hence e is forcibly $aK_2 + bK_1$-free. Since $d \succeq e$ and $\mathcal{F}$ is a dominance monotone set, some element of $\mathcal{F}$ must be an induced subgraph of some disjoint union of cycles having at most $3a + 2b - 1$ vertices. □

We now characterize the dominance monotone sets of cardinality 2, as follows.

Theorem 2.4.2 *A set $\mathcal{F}$ of two graphs is dominance monotone if and only if one of the following is true:*

(i) *$\mathcal{F}$ contains one of K_1, $2K_1$, or K_2;*

(ii) *$\mathcal{F}$ is one of $\{K_2 + K_1, P_3\}$, $\{K_2 + K_1, C_4\}$, or $\{2K_2, P_3\}$.*

Proof. Sufficiency of the conditions (i) and (ii) follows from Proposition 2.3.4. We now prove their necessity.

To begin, we show that the only dominance monotone pairs containing P_3 or $K_2 + K_1$ are the ones indicated, as follows: If $\mathcal{F} = \{P_3, B\}$ is dominance monotone, then since $(211, 1111)$ should not be a counterexample pair, B must be induced in $2K_2$; every such graph B yields one of the pairs from Theorem 2.4.2. If instead the dominance monotone set is $\{K_2 + K_1, B\}$, then for $(3221, 2222)$ to not be a counterexample pair, B must be induced in C_4; every possibility for B yields a set from the theorem statement.

Suppose now that $\mathcal{F} = \{A, B\}$ is a dominance monotone set in which A and B each have at least 3 vertices. Further assume that neither A nor B is an induced subgraph of the other; otherwise, if A is induced in B, the $\mathcal{F}$-free graphs are precisely the A-free graphs, and Theorem 2.3.5 implies that $\mathcal{F}$ is dominance monotone if and only if the condition (i) holds.

By Theorem 2.3.2, we may assume without loss of generality that $\Delta(A) \leq 1$. Hence A has the form $aK_2 + bK_1$ for some nonnegative a and b. Since A has at least three vertices, if $a = 0$ then $b \geq 3$, and if $a = 1$ then $b \geq 1$.

Recall from Corollary 2.3.3 that some element of $\mathcal{F}$ either has a dominating vertex or is regular with degree its order minus 2. This element cannot be A; otherwise, as in the proof of Theorem 2.3.5, A would have two or fewer vertices, contrary to our assumption. Hence B is the element of $\mathcal{F}$ with this property. Lemma 2.4.1 implies that B must also be induced in a disjoint union of cycles; thus $\Delta(B) \leq 2$. These several requirements on B imply that it is one of P_3, K_3, or C_4. The case $B = P_3$ was handled previously. If B is K_3 or C_4, then $(222, 2211)$ or $(32221, 2^5)$, respectively, is a counterexample pair. □

2.5 Dominance monotone triples

In this section we characterize the dominance monotone sets of cardinality 3. In the following, the *diamond* is the graph $K_4 - e$ for an edge e.

Theorem 2.5.1 *A set $\mathcal{F}$ of three graphs is dominance monotone if and only if one of the following is true:*

(i) *$\mathcal{F}$ contains a dominance monotone singleton or pair;*

(ii) $\mathcal{F}$ *is one of* $\{2K_2, P_4, \text{diamond}\}$, $\{K_2 + 2K_1, P_4, C_4\}$, $\{2K_2, P_4, C_4\}$, $\{2K_2, C_4, C_5\}$.

The proof will occupy the remainder of this section. We first show the sufficiency of the conditions (i) and (ii). Condition (i) and $\mathcal{F} = \{2K_2, P_4, C_4\}$ both imply that $\mathcal{F}$ is dominance monotone by Proposition 2.3.4. That $\{2K_2, C_4, C_5\}$ is dominance monotone was shown by Merris [6].

Proposition 2.5.2 *The triples* $\{2K_2, P_4, \text{diamond}\}$ *and* $\{K_2 + 2K_1, P_4, C_4\}$ *are dominance monotone.*

Proof. Since each of the two sets contains complements of the other set's graphs, by Theorem 2.3.1 it suffices to show that $\mathcal{F} = \{K_2 + 2K_1, P_4, C_4\}$ is a dominance monotone set.

Assume that $d \succeq e$ and e is forcibly $\mathcal{F}$-free. If e is also forcibly $2K_2$-free, then e is a threshold sequence, and it is vacuously true that d is forcibly $\mathcal{F}$-free. Suppose instead that e is not forcibly $2K_2$-free.

We claim that any graph that is $\mathcal{F}$-free and contains $2K_2$ as an induced subgraph may have its vertices partitioned into two cliques and a set of dominating vertices. Indeed, consider such a graph G, and suppose that the edges of some induced subgraph isomorphic to $2K_2$ are pq and rs.

Let Q_1 be a maximal clique of G containing p and q, and let Q_2 be a maximal clique of G containing r and s.

We claim that no vertex lies outside Q_1 and Q_2. Suppose to the contrary that $v \in V(G) - Q_1 \cup Q_2$. Let $Q_1' = Q_1 - Q_2$ and $Q_2' = Q_2 - Q_1$ (see Figure 5). If Q_1' contains more than one non-neighbor of v, say v_1, u_1, then for any non-neighbor v_2 in Q_2', the set $\{u_1, v_1, v, v_2\}$ induces $K_2 + 2K_1$, a contradiction. Suppose that v is adjacent to all vertices in Q_2'. Since $v \notin Q_2$ and Q_2 is maximal as a clique,

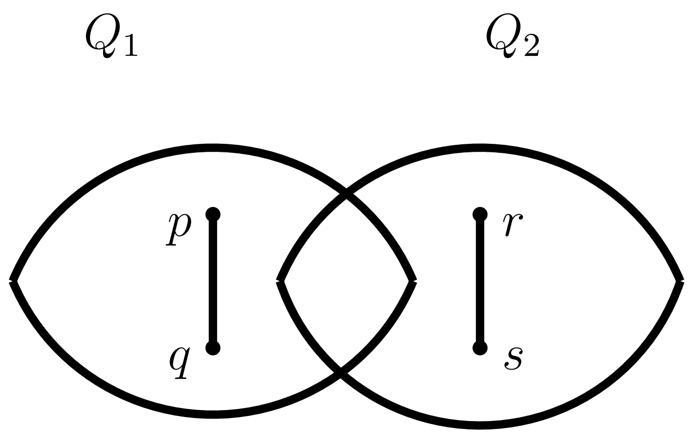

Figure 5. The sets Q_1, Q_2 in G.

v has non-neighbor w in $Q_1 \cap Q_2$; however, then the set $\{v, r, w, u_1\}$ induces a P_4 in G, a contradiction. Hence v is adjacent to all but possibly one vertex of Q_1'. Similarly, v is adjacent to all but possibly one vertex of Q_2'. Therefore, we may assume without loss of generality that v is adjacent to p in Q_1' and r in Q_2'. If v has any non-neighbor v_1 in Q_1', then the set $\{v_1, p, v, r\}$ induces P_4, a contradiction. Thus v is adjacent to every vertex in Q_1' and similarly, to every vertex in Q_2'.

Since v is adjacent to every vertex in $Q_1' \cup Q_2'$ but v does not belong to $Q_1 \cup Q_2$, there is a vertex $u \in Q_1 \cap Q_2$ that is not adjacent to v. However, the set $\{u, v, p, r\}$ then induces C_4, a contradiction. Hence no such vertex v exists; $V(G) = Q_1 \cup Q_2$.

Now suppose that the degree sequence of G is forcibly $\{K_2 + 2K_1, P_4, C_4\}$-free. We claim that $Q_1' = \{p, q\}$ and $Q_2' = \{r, s\}$. Suppose to the contrary that Q_1' has another vertex x. Deleting the edges pq and rs and adding the edges pr and qs yields another realization of the degree sequence of G. However, the vertices $\{r, p, x, q\}$ induce a P_4 in G, which is a contradiction to e being forcibly $\mathcal{F}$-free.

Hence e has the general form $e = (k+3)^k (k+1)^4$ where k is a nonnegative integer. Note that the first k terms of e correspond to dominating vertices in G and hence are maximal for the length of this degree sequence. Thus, if $d \succeq e$, then d can only differ from e in the last four terms. It follows from Muirhead's

Lemma and inspection that d is the threshold sequence $(k+3)^k(k+2)^1(k+1)^2k^1$, which has a unique realization obtained when k dominating vertices are added to P_3+K_1. This graph is forcibly K_2+2K_1-free, so $\{K_2+2K_1, P_4, C_4\}$ is dominance monotone. $\square$

We now prove the necessity of Conditions (i) and (ii) in Theorem 2.5.1. Suppose that $\mathcal{F} = \{A, B, C\}$ is a dominance monotone set.

If C contains A or B as an induced subgraph, then the $\mathcal{F}$-free graphs are precisely the $\{A, B\}$-free graphs, and $\{A, B\}$ is a dominance monotone pair (possibly containing a dominance monotone singleton), as in (i). Assume henceforth that none of A, B, C is an induced subgraph of another; the order of each of A, B, C is then at least 3.

By Theorem 2.3.2, we assume without loss of generality that $A = aK_2+bK_1$ for some integers a, b. By Corollary 2.3.3, $\mathcal{F}$ contains either a graph with a dominating vertex or a graph that is regular of degree 2 less than its order. As in the previous section we conclude that this graph is not A; without loss of generality we assume it is B.

By Lemma 2.4.1, $\mathcal{F}$ contains an induced subgraph of a disjoint union of cycles on at most $3a + 2b - 1$ vertices; we saw there that such a graph cannot contain $aK_2 + bK_1$ as an induced subgraph. Hence either B or C is an induced subgraph of a disjoint union of cycles.

If B is induced in a disjoint union of cycles, then $\Delta(B) \leq 2$. Because $|V(B)| \geq 3$ and B has a dominating vertex or is $(|V(B)| - 2)$-regular, B must be P_3 or K_3 or C_4. We will handle these possibilities now, along with a few other cases that will be useful in the future.

Lemma 2.5.3 *Every dominance monotone triple containing P_3 or K_3 or K_2+K_1 contains a dominance monotone singleton or pair. Every dominance monotone* *triple containing C_4 either contains a dominance monotone singleton or pair or is one of $\{K_2 + 2K_1, P_4, C_4\}, \{2K_2, P_4, C_4\}, \{2K_2, C_4, C_5\}$. Every dominance monotone triple containing $2K_2$ and P_4 either contains a dominance monotone singleton or pair or is $\{2K_2, P_4, C_4\}$ or $\{2K_2, P_4, \text{diamond}\}$.*

Proof. Let $\mathcal{F} = \{A, B, C\}$ be an arbitrary dominance monotone set. By Theorem 2.3.2 we may assume that $A = aK_2 + bK_1$ for nonnegative integers.

If $B = P_3$, then since $(211, 1111)$ is not a counterexample pair, either A or C must be induced in $2K_2$. By Theorem 2.4.2 this graph and B then form a dominance monotone pair.

If $A = K_2 + K_1$, then since $(3221, 2222)$ is not a counterexample pair, either B or C must be induced in C_4. By Theorem 2.4.2 this graph and A then form a dominance monotone pair.

If $B = K_3$, then since $(3221, 2222)$ is not a counterexample pair, either A or C is an induced subgraph of C_4. By Theorem 2.4.2 the set $\mathcal{F}$ will contain a dominance monotone singleton or pair unless $C = C_4$. With $C = C_4$, since $(32221, 2^5)$ is not a counterexample pair, $\mathcal{F}$ contains an induced subgraph of C_5, which must be A. Since A has at least three vertices, we conclude that $A = K_2 + K_1$; then $\mathcal{F}$ contains the dominance monotone pair $\{K_2 + K_1, C_4\}$.

If $B = C_4$, then since $(32221, 2^5)$ is not a counterexample pair, $\mathcal{F}$ contains an induced subgraph of C_5. If A is this subgraph, then either A has fewer than three vertices (in which case $\mathcal{F}$ contains a dominance monotone singleton, satisfying our claim), or $A = K_2 + K_1$, which was discussed previously. Assume that C is induced in C_5. The cases where C is P_3 or $K_2 + K_1$ or a graph with fewer than three vertices lead to $\mathcal{F}$ containing a dominance monotone singleton or pair, so we may assume that $C = P_4$ or $C = C_5$.

If $B = C_4$ and $C = C_5$, then since $(2222, 22211)$ is not a counterexample pair for $\mathcal{F}$, the graph A must be induced in K_2+K_3 or P_5 and hence is induced in $2K_2$. If A is K_2+K_1 or has fewer than three vertices, then $\mathcal{F}$ contains a dominance monotone singleton or pair. Otherwise, $A = 2K_2$, and $\mathcal{F} = \{2K_2, C_4, C_5\}$.

If $B = C_4$ and $C = P_4$, then since $(2211, 21111)$ is not a counterexample pair, A is induced in $P_3 + K_2$. If A has three or fewer vertices or is $K_2 + K_1$, then $\mathcal{F}$ contains a dominance monotone singleton or pair; otherwise, A is one of $3K_1, 2K_2, K_2 + 2K_1$. If $A = 3K_1$, we have $(43221, 42222)$ as a counterexample pair, a contradiction. When $A = 2K_2$ we have $\mathcal{F} = \{2K_2, P_4, C_4\}$, and when $A = K_2 + 2K_1$, we have $\mathcal{F} = \{K_2 + 2K_1, P_4, C_4\}$.

If $A = 2K_2$ and $C = P_4$, then consider the pair (d, e), where $d = 43322$ (the degree sequence of $K_1 \vee P_4$) and $e = 33332$, which has a unique realization that is obtained by subdividing an edge of K_4. Observe that the realization of e contains no induced $2K_2$ or P_4. Since $\mathcal{F}$ is dominance monotone, B must be induced in the unique realization of 33332. Since B either is $(|V(B)| - 2)$-regular or has a dominating vertex, we see that either B is C_4 or B is P_3 (which was discussed above) or $K_1 \vee (K_2 + K_1)$ or the diamond graph. The possibility $B = K_1 \vee (K_2 + K_1)$ is eliminated by the counterexample pair $(3221, 2222)$, so we conclude that $\mathcal{F}$ is either $\{2K_2, C_4, P_4\}$ or $\{2K_2, P_4, \text{diamond}\}$. $\square$

Assume henceforth that the dominance monotone triple $\mathcal{F}$ contains none of P_3 $K_2 + K_1$, K_3, or C_4, and that it does not contain the pair $\{2K_2, P_4\}$ as a subset. Having determined the dominance monotone triples where $\Delta(B) \leq 2$, we will assume in the remainder of the proof that $\Delta(B) \geq 3$ and that C is induced in a disjoint union of cycles on at most $3a + 2b - 1$ vertices.

To help further restrict our search for dominance monotone triples, we present some further requirements for the set $\mathcal{F}$.

Lemma 2.5.4 *If $\mathcal{F}$ is a dominance monotone set containing $aK_2 + bK_1$ or $K_1 \vee$*

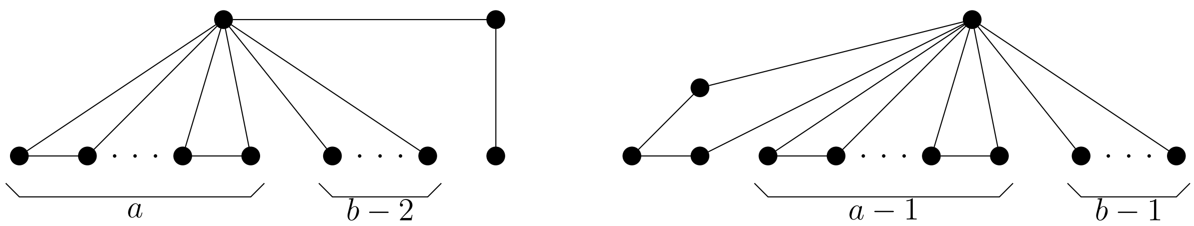

Figure 6. The two realizations of $(b+2a-1)^1 2^{2a+1} 1^{b-1}$.

$(aK_2 + bK_1)$ for $b \geq 1$ (and $b \geq 3$ if $a = 0$), then $\mathcal{F}$ contains an induced subgraph of a graph obtained by subdividing one edge of $K_1 \vee (aK_2 + (b-1)K_1)$; any such induced subgraph is $\{aK_2 + bK_1\}$-free.

Proof. Consider the degree sequences $d = (b+2a)^1 2^{2a} 1^b$ and $e = (b+2a-1)^1 2^{2a+1} 1^{b-1}$. Clearly d majorizes e. Observe that the unique realization of d is a graph isomorphic to $K_1 \vee (aK_2 + bK_1)$.

We show that e has at most two realizations. In any realization G of e, a vertex of maximum degree has one non-neighbor. If this non-neighbor has degree 1 (which can only happen if $b \geq 2$), then deleting a vertex of maximum degree yields a graph with degree sequence $1^{2a+2} 0^{b-2}$, which has a unique realization in $(a+1)K_2 + (b-2)K_1$. Thus, G is the graph obtained from $K_1 \vee (aK_2 + (b-1)K_1)$ by subdividing a pendant edge, as in the graph on the left in Figure 6.

If G is a realization of e in which a vertex v of maximum degree has a non-neighbor with degree 2 (which can only happen if $a \geq 1$, since the degree-2 vertex cannot have neighbors among the vertices of degree 1), then deleting v yields a graph with degree sequence $2^1 1^{2a} 0^{b-1}$, which has a unique realization in $P_3 + (a-1)K_2 + (b-1)K_1$. Thus G is the graph obtained from $K_1 \vee (aK_2 + (b-1)K_1)$ by subdividing an edge of a triangle, as in the graph on the right in Figure 6.

Inspection shows that neither realization of e contains $aK_2 + bK_1$ as an induced subgraph, so e is forcibly $\{aK_2 + bK_1\}$-free. Since $d \succeq e$ and $\mathcal{F}$ is dominance monotone, B must be induced in one of the graphs in Figure 6. $\square$

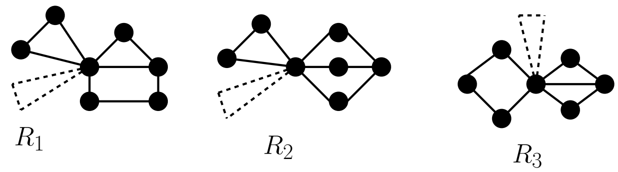

Figure 7. All possible realizations of $\varepsilon = (2a-1)^1 3^1 2^{2a-1}$. In realizations R_1 and R_2, $a-2$ triangles are attached to a vertex of degree 3 in a realization of 33222. In realization R_3, $a-3$ triangles and a single C_4 are attached to a dominating vertex in the diamond.

Lemma 2.5.5 *If $\mathcal{F}$ is a dominance monotone set containing aK_2 or $K_1 \vee aK_2$, then $\mathcal{F}$ contains an induced subgraph of at least one of the realizations of $\varepsilon = (2a-1)^1 3^1 2^{2a-1}$ (see Figure 7); any such induced subgraph is $\{aK_2\}$-free.*

If H is a graph that is induced in one of the realizations of ε and H has a dominating vertex, then H is one of the following:

- $K_1 \vee (pK_2 + qK_1)$, *where* $p \leq a-1$ *(and* $p = a-1$ *only if* $q \leq 1$*), and* $p + q \leq a+1$*;*
- $K_1 \vee (P_3 + pK_2 + qK_1)$, *where* $p \leq a-3$ *and* $p+q \leq a-1$ *(this possibility only arises if* $a \geq 3$*).*

If H is an induced subgraph of a realization of ε with $\Delta(H) \leq 2$, then it satisfies the following:

- *if* $\Delta(H) \leq 1$*, then* $H = sK_2 + tK_1$*, where* $s \leq a-1$ *(and* $s = a-1$ *only if* $t \leq 1$*) and* $s + t \leq a+1$.
- *if* $\Delta(H) = 2$*, then* H *is one of the following:*
  - K_3 *or* C_4*;*
  - $P_4 + cK_2 + dK_1$*, where* $c + d \leq a-2$*;*
  - $P_3 + cK_2 + dK_1$ *where* $c+d \leq a-1$ *(where* $c+d = a-1$ *only if* $a \geq 3$*);*

– $2P_3 + cK_2 + dK_1$ *where* $c + d \leq a - 3$ *(this possibility only arises if* $a \geq 3$*).*

Proof. Given that $\mathcal{F}$ contains aK_2 or $K_1 \vee aK_2$, consider the pair $((2a)^1 2^{2a}, \varepsilon)$. Clearly, $d_1 \succeq \varepsilon$, and d_1 is not forcibly $\mathcal{F}$-free, since its unique realization is the graph $K_1 \vee aK_2$. Since $\mathcal{F}$ is dominance monotone, this pair of degree sequences is not a counterexample pair, so $\mathcal{F}$ contains an induced subgraph of a realization of ε.

To see that the induced subgraph is not aK_2 or $K_1 \vee aK_2$ when $a \geq 2$, it suffices to realize that the maximum degree vertex in a realization of ε cannot belong to an induced copy of aK_2, for it is adjacent to all but one vertex. Thus an induced copy of aK_2 must contain all the other vertices, which is impossible since the degree-3 vertex is adjacent to at least two vertices of degree 2.

In any realization G of ε the vertex u of maximum degree is adjacent to all but one vertex v of G. If v has degree 2 in G, then the graph $G - u$ has degree sequence $2^2 1^{2a-2}$ and hence is isomorphic to either $P_4 + (a-2)K_2$ or $2P_3 + (a-3)K_2$, and the graph G is of the type shown in realizations R_1 or R_3 in Figure 7. If instead v has degree 3 in G, then the degree sequence of $G - u$ is $3^1 1^{2a-1}$ and hence $G - u$ is $K_{1,3} + (a-2)K_2$, leading G to be of the form shown in R_2 in Figure 7.

Inspection of the realizations of ε yields the possibilities for H if H is induced in a realization of ε and has a dominating vertex or has maximum degree at most 2. □

With these conditions on $\mathcal{F}$ established, we organize the rest of the proof of the necessity of (i) and (ii) in Theorem 2.5.1 by recalling that B has a dominating vertex or is $(|V(B)| - 2)$-regular. We will handle the two possibilities for the structure of B in separate subsections.

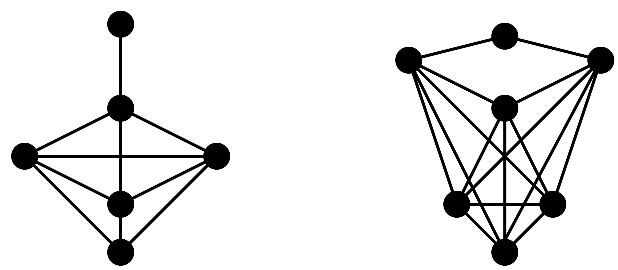

Figure 8. The unique realizations of $4^4 3^1 1^1$ and $5^6 2^1$.

2.5.1 Case: B has a dominating vertex

We begin with two helpful lemmas on dominance monotone sets containing graphs of certain types.

Lemma 2.5.6 *If $\mathcal{F}$ is a dominance monotone set containing P_4, then $\mathcal{F}$ contains an induced subgraph of the graph obtained by subdividing an edge of K_6.*

Proof. Since $\mathcal{F}$ is dominance monotone, the degree sequences $(6^1 5^4 4^1 2^1, 5^6 2^1)$ do not form a counterexample pair; note that a realization of $6^1 5^4 4^1 2^1$ is obtained by adding a dominating vertex to $4^4 3^1 1^1$, which has a realization inducing P_4 as shown in Figure 8, and the unique realization of $5^6 2^1$ is obtained by subdividing an edge of K_6. $\square$

Lemma 2.5.7 *Let $\mathcal{F}$ be a dominance monotone set containing $P_3 + pK_2 + qK_1$ or $K_1 \vee (P_3 + pK_2 + qK_1)$. If $\mathcal{F}$ contains $K_1 \vee P_3$ (i.e., if $q = p = 0$), then $\mathcal{F}$ contain an induced subgraph of C_5. If $q = 0$ and $p \geq 1$, then $\mathcal{F}$ must contain an induced subgraph H of at least one of the realizations of $e_2 = (2p+2)^1 4^1 2^{2p+2}$ in Figure 9. If $q \geq 1$ then $\mathcal{F}$ must contain an induced subgraph of $K_1 \vee ((p+2)K_2 + (q-1)K_1)$.*

If $p + q \geq 1$, then the induced subgraphs described are $\{P_3 + pK_2 + qK_1\}$-free. Moreover, if $q = 0$ and $p \geq 1$, then H satisfies the following:

- *if $\Delta(H) \leq 1$, then $H = sK_2 + tK_1$, where $s \leq p$ and $s + t \leq p + 3$.*

- *if $\Delta(H) = 2$, then H is one of the following.*

– K_3, C_4, *or* P_4;

– $P_3 + sK_2 + tK_1$ *for some* s, t *such that* $s \leq p - 1$ *and* $s + t \leq p + 1$;

– $2P_3 + sK_2 + tK_1$ *for some* s, t *such that* $s + t \leq p - 2$.

Proof. When $q = p = 0$, the set $\mathcal{F}$ contains $K_1 \vee P_3$, since $\mathcal{F}$ does not contain P_3. Since $(3322, 22222)$ is not a counterexample pair, $\mathcal{F}$ must contain an induced subgraph of C_5.

When $q \neq 0$, it suffices to realize that (d, e_1) is not a counterexample pair, where $d = (2p+q+3)^1 3^1 2^{2p+2} 1^q$ (the degree sequence of $K_1 \vee P_3 + pK_2 + qK_1$) and $e_1 = (2p + q + 3)^1 2^{2p+4} 1^{q-1}$ (the degree sequence of $K_1 \vee ((p + 2)K_2 + (q - 1)K_1)$.

When $q = 0$ and $p \geq 1$, consider the pair (d, e_2) where d is as above and $e_2 = (2p + 2)^1 4^1 2^{2p+2}$; since this is not a counterexample pair, $\mathcal{F}$ contains an induced subgraph of a realization of e_2. We determine the realizations of e_2 as follows. Let H be a realization of e_2. Let u and v be the vertices of maximum degree and degree 4, respectively. Observe that u is adjacent to all but one of the other vertices in H. If u is not adjacent to v, then $H - u$ has degree sequence $4^1 1^{2p+2}$, which is uniquely realized by $K_{1,4} + (p - 1)K_2$, and H therefore has the form shown in the first graph in Figure 9. If u is adjacent to v, then $H - u$ has degree sequence $3^1 2^1 1^{2p+1}$, which has realizations $T + (p - 1)K_2$, where T is the tree obtained by attaching two pendant vertices to an endpoint of P_3, and $K_{1,3} + P_3 + (p - 2)K_2$ (which is possible only if $p \geq 2$). In these cases the graph H has a form shown in the second and third graphs, respectively, in Figure 9.

That the realizations of e_1 and of e_2 are all $\{P_3 + pK_2 + qK_1\}$-free when $p + q \geq 1$ can be easily verified by inspection. Inspection also confirms the stated conditions on H when $\Delta(H) \leq 2$. □

With our preliminary lemmas established, recall that $A = aK_2 + bK_1$, that B has a dominating vertex, and that C is induced in a disjoint union of cycles on

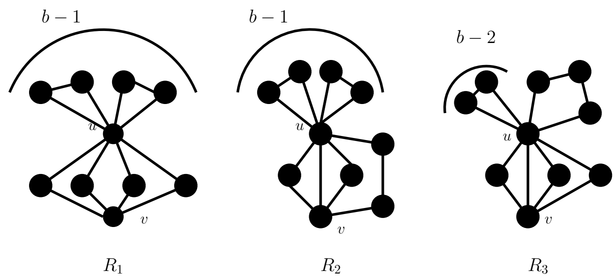

Figure 9. All possible realizations of $e_2 = (2p+2)^1 4^1 2^{2p+2}$.

at most $3a + 2b - 1$ vertices. We proceed by subcases on the number of isolated vertices in A.

Subcase 1: $b = 0$**.**

Here $A = aK_2$, where $a \geq 2$ by our assumption that A has at least three vertices, and C is induced in a disjoint union of cycles on at most $3a - 1$ vertices. By Lemma 2.5.5, $\mathcal{F}$ contains an induced subgraph in at least one of the realizations of $\varepsilon = (2a-1)^1 3^1 2^{2a-1}$, and this graph is not A. Therefore, either B or C is induced in at least one realization of ε.

Suppose first that B is induced in at least one realization of ε. By Lemma 2.5.5 B is equal to either $K_1 \vee (pK_2 + qK_1)$ or $K_1 \vee (P_3 + pK_2 + qK_1)$, where $p + q$ is bounded according to the values of p and q. We will consider each of these possibilities for B in turn.

Case: $B = K_1 \vee pK_2$*, where* $p \leq a - 1$. We may assume that $p \geq 2$, since B is assumed not to be K_3. Moreover, by Lemma 2.5.5, $\mathcal{F}$ must contain an induced subgraph in at least one of the realizations of $\varepsilon' = (2p-1)^1 3^1 2^{2p-1}$; each such realization is $\{pK_2\}$-free and hence $\{A, B\}$-free, so C is induced in some realization of ε'. If $\Delta(C) \leq 1$, then $C = sK_2 + tK_1$ for some integers s, t bounded as in Lemma 2.5.5. If $t = 0$, then C is induced in A, contrary to our assumption, so $C = sK_2 + tK_1$ where $t \geq 1$ and $s \leq p - 1$. Thus, by Lemma 2.5.4, $\mathcal{F}$ must contain an induced subgraph of a graph obtained by subdividing one edge of

$K_1 \vee (sK_2 + (t-1)K_1)$, but none of A, B, or C is such an induced subgraph, a contradiction.

If $\Delta(C) = 2$, Lemma 2.5.5 lists all graphs that C can be. Our assumptions exclude the possibilities of C being one K_3 or C_4. If $C = P_4 + s'K_2 + t'K_1$, where $s' + t' \le p - 2$, consider the pairs (d, e_1) and (d, e_2), where

$$\begin{aligned} d &= (2s' + t' + 4)^1 3^2 2^{2s'+2} 1^{t'}, \\ e_1 &= (2s' + 3)^1 4^1 3^1 2^{2s'+2}, \\ e_2 &= (2s' + t' + 3)^1 3^2 2^{2s'+3} 1^{t'-1}. \end{aligned}$$

Note that d is the degree sequence of $K_1 \vee C$. If $t' = 0$, then e_1 is forcibly $\{A\}$-free because otherwise $2a \le 2s' + 5$, yielding $a - 2 \le s' \le p - 2 \le a - 3$, a contradiction. The sequence e_1 is also forcibly $\{B\}$-free since otherwise $2p + 1 \le 2s' + 5 \le 2(p-2) + 5$, implying that B is the realization of e_1, a contradiction since B has a dominating vertex. Finally, any realization of e_1 has exactly one vertex more than C; if $s' \ge 1$, then deleting any vertex from such a realization leaves a subgraph with maximum degree at least 3, so e_1 is forcibly $\{C\}$-free. Thus (d, e_1) is a counterexample pair if $t' = 0$ unless $s' = 0$ and hence $C = P_4$. In this case, the result of Lemma 2.5.6 requires that $\mathcal{F}$ contain an induced subgraph of an edge-subdivided K_6, which true of none of aK_2, $K_1 \vee pK_2$, or P_4, a contradiction.

If $t' \ge 1$, then e_2 is forcibly $\{A\}$-free, since otherwise

$$2a \le 2s' + t' + 5 < 2(s' + t') + 5 \le 2(p-2) + 5 \le 2a - 1,$$

a contradiction. The sequence e_2 is forcibly $\{B\}$-free, since otherwise

$$2p + 1 \le 2s' + t' + 5 < 2(s' + t') + 5 \le 2(p-2) + 5 = 2p + 1,$$

a contradiction. The sequence e_2 is forcibly $\{C\}$-free because any realization of e_2 has exactly one more vertex than C, but deleting a single vertex from such a realization cannot leave t' isolated vertices. Thus (d, e_2) is a counterexample pair, and this contradiction concludes the possibility that $C = P_4 + s'K_2 + t'K_1$.

Suppose instead that, as in Lemma 2.5.5, $C = P_3 + s'K_2 + t'K_1$, where $s' + t' \leq p - 1$ and $s' + t' = p - 1$ only if $p \geq 3$. Consider the pairs (d, e_1) and (d, e_2), where

$$\begin{aligned} d &= (2s' + t' + 3)^1 3^1 2^{2s'+2} 1^{t'}, \\ e_1 &= (2s' + 2)^1 4^1 2^{2s'+2}, \\ e_2 &= (2s' + t' + 2)^1 3^1 2^{2s'+3} 1^{t'-1}. \end{aligned} \tag{1}$$

The arguments here proceed in much the same way as in the last paragraph, except in the following few ways. To conclude that e_1 is forcibly $\{A\}$-free we also note that if $2a \leq 2s' + 4$, then $s' = p - 1 = a - 2$, from which it follows that A is a realization of e_1, a contradiction. To conclude that e_1 is forcibly $\{B\}$-free we note that if $2p + 1 \leq 2s' + 4$, then B can be obtained by deleting one vertex from a realization of e_1, and no such vertex deletion yields B. To conclude that e_1 is forcibly $\{C\}$-free we may assume that $s + t \geq 1$, since by assumption $C \neq P_3$. To conclude that e_2 is forcibly $\{B\}$-free, we note that if $2p + 1 \leq 2s' + t' + 4$, then B is a realization of e_2, a contradiction.

The above contradictions imply, by Lemma 2.5.5, that $C = 2P_3 + s'K_2 + t'K_1$, where $s' + t' \leq p - 3$ and $s' + t' = p - 3$ only if $p \geq 3$. Consider the pairs (d, e_1) and (d, e_2), where

$$\begin{aligned} d &= (2s' + t' + 6)^1 3^2 2^{2s'+4} 1^{t'}, \\ e_1 &= (2s' + 5)^1 4^1 3^1 2^{2s'+4}, \\ e_2 &= (2s' + t' + 5)^1 3^2 2^{2s'+5} 1^{t'-1}. \end{aligned}$$

The arguments showing that (d, e_1) and (d, e_2) are counterexample pairs in the cases $t' = 0$ and $t' \geq 1$, respectively, are again analogous to those in the case $C = P_4 + s'K_2 + t'K_1$ above. We omit the details and conclude that this possibility for C also ends in contradiction.

Case: $B = K_1 \vee (pK_2 + qK_1)$, *where* $p \leq a-1$ *(and* $p = a-1$ *only if* $q \leq 1$*) and* $p+q \leq a+1$. By the previous case, we may assume that $q \geq 1$, and since B is not P_3, we assume that $q \geq 3$ if $p = 0$. Then by Lemma 2.5.4, either A or C must also be an induced subgraph of a graph obtained by subdividing an edge of $K_1 \vee (pK_2 + (q-1)K_1)$. Since A is not an induced subgraph, C is, besides being induced in the disjoint union of cycles having at most $3a-1$ vertices. If $\Delta(C) \leq 1$, then $C = sK_2 + tK_1$ for some s, t such that $s \leq p+1 \leq a$ and $s+t \leq p+q \leq a+1$. If $t = 0$ then C is induced in A, and if $s = 0$ then C is induced in B, contrary to our assumption, so we see that $s, t \neq 0$ and $t \geq 2$ when $s = 1$ since C has at least 3 vertices and is not $K_2 + K_1$. Thus, by Lemma 2.5.4, $\mathcal{F}$ must contain an induced subgraph of a graph obtained by subdividing an edge of $K_1 \vee (sK_2 + (t-1)K_1)$, but such a graph is $\{A, B, C\}$-free, a contradiction. If $\Delta(C) = 2$, then since C is not P_3 or K_3 or C_4, we see that C is either $K_3 + K_1$ or P_4 or $P_3 + s'K_2 + t'K_1$ where $s' + t' \geq 1$. For $C = K_3 + K_1$, we find that $(32^31, 2^5)$ is a counterexample pair, a contradiction. When $C = P_4$, we find that $(6^1 5^4 4^1 2^1, 5^6 2^1)$ is a counterexample pair, a contradiction. For $C = P_3 + s'K_2 + t'K_1$, the degree sequences in (1) form counterexample pairs for analogous reasons.

Case: $B = K_1 \vee (P_3 + pK_2 + qK_1)$, *where* $p \leq a-3$ *and* $p+q \leq a-1$. As in Lemma 2.5.5, this case requires that $a \geq 3$.

Assume now that p or q is nonzero. By Lemma 2.5.7, $\mathcal{F}$ contains an induced subgraph of $K_1 \vee ((p+2)K_2 + (q-1)K_1)$ if $q \neq 0$, or an induced subgraph of a realization of $(2p+2)^1 4^1 2^{2p+2}$ if $q = 0$. Neither A nor B can satisfy these requirements, so C is the desired induced subgraph, and Lemma 2.5.7 implies that C is P_4 or $sK_2 + tK_1$ or $P_3 + sK_2 + tK_1$ or $2P_3 + sK_2 + tK_1$ for suitable s, t.

If $C = P_4$, then the pair $(2211, 21111)$ is a counterexample pair, a contradiction.

If $C = sK_2 + tK_1$ (where Lemma 2.5.7 tells us $s \leq p+2$), then Lemmas 2.5.4 and 2.5.5 imply that $\mathcal{F}$ contains an $\{C\}$-free graph H whose largest induced matching has size at most $s+1$. Since $s \leq p+2 \leq a-1$, the graph H does not contain A as an induced subgraph. Since B contains the diamond as an induced subgraph, H is $\{B\}$-free as well unless H is contained in a graph of the form R_3 in Figure 7 having at most $s-3$ triangles, forcing $p \leq s-3 \leq p-1$, a contradiction.

If $C = P_3+sK_2+tK_1$ (where Lemma 2.5.7 tells us $s \leq p-1$), then Lemma 2.5.7 implies that $\mathcal{F}$ contains a $\{C\}$-free graph H that is induced in $K_1 \vee ((s+2)K_2 + (t-1)K_1)$ or in a realization of $(2s+2)^1 4^1 2^{2s+2}$. Since all such graphs have largest induced matchings of order at most $s+2$, and $s+2 \leq p+1 \leq a-2$, the graph H is $\{A\}$-free. Since $\mathcal{F}$ is dominance monotone, H must contain B an induced subgraph. Now $K_1 \vee ((s+2)K_2 + (t-1)K_1)$ contains no induced $K_1 \vee P_3$, as B does, so B must be induced in a realization of $(2s+2)^1 4^1 2^{2s+2}$. Note that only the realizations R_2 and R_3 in Figure 9 contain $K_1 \vee P_3$ as an induced subgraph. Assume that $p+q \geq 1$. The unique vertex of B with degree at least 4 must be the vertex u of maximum degree in R_2 or R_3, and the unique vertex of degree 3 in B is the vertex of second-highest degree in R_2 or R_3. In either realization, the remaining vertices adjacent to u do not yield $pK_2 + qK_1$ as an induced subgraph, a contradiction, since B is induced in H.

If $C = 2P_3 + sK_2 + tK_1$, (where Lemma 2.5.7 tells us $s+t \leq p-2$), then for $t \neq 0$ we claim that $\mathcal{F}$ must contain a $\{C\}$-free graph H that is induced in $K_1 \vee (P_3 + (s+2)K_2 + (t-1)K_1)$; for otherwise (d,e) would be a counterexample pair, where $d = (2s+t+6)^1 3^2 2^{2s+4} 1^t$ (the degree sequence of $K_1 \vee (2P_3+2K_2+tK_1)$) and $e = (2s+t+6)^1 3^1 2^{2s+6} 1^{t-1}$, since the unique realization of e is $K_1 \vee (P_3 + (s+2)K_2 + (t-1)K_1)$, which contains only one induced P_3. It is not hard to see that in this graph the largest induced matching has order at most $s+3 \leq p \leq a-3$, so this graph is also $\{A\}$-free and hence must contain B as an induced subgraph, since $\mathcal{F}$ is dominance monotone. However, since the realization contains exactly one diamond, this leaves only $s+t+1 \leq p-1$ vertices to obtain an induced pK_2+qK_1, which is not possible, a contradiction. If $t = 0$, then $\mathcal{F}$ must contain a $\{C\}$-free graph H' that is induced in $P_3 + (s+2)K_2$, the unique realization of $e' = 21^{2s+6}$, for otherwise (d', e') would be a counterexample pair, where $d' = 2^2 1^{2s+4}$, since the realization of e' contains only one P_3. The largest induced matching in $P_3+(s+2)K_2$ is at most $s+3 \leq p+1 \leq a-2$. Thus A is not induced in any realization of e' and since B has maximum degree at least 3, B is not induced either, a contradiction.

The contradictions above show that B is not induced in any realization of ε, so C must be instead. Recall that $\Delta(C) \leq 2$.

If $\Delta(C) \leq 1$, then $C = sK_2 + tK_1$ and by Lemma 2.5.5, $s + t \leq a + 1$, and $s \leq a - 2$ if $t > 1$; otherwise $s \leq a - 1$. It $t = 0$, then C is induced in A contrary to our assumption, so $C = sK_2 + tK_1$ where $t \geq 1$ and $t \geq 3$ when $s = 0$. If both s and t are equal 1, then Lemma 2.5.3 applies, and $\mathcal{F}$ contains a dominance monotone singleton or pair. In any other case, by Lemma 2.5.4 $\mathcal{F}$ must contain a $\{C\}$-free induced subgraph H of a graph obtained by subdividing one edge of $K_1 \vee (sK_2 + (t-1)K_1)$. A maximum induced matching in H has at most s edges if $t = 1$ and $s + 1$ edges if $t > 1$. Since $s \leq a - 1$, the graph A is not induced in H. Then B is induced in H, and since B is assumed not to be P_3 or K_3, we have $B = K_1 \vee (s'K_2 + t'K_1)$ for s', t' such that $s' \leq s$ and $s' + t' \leq s + t$. If $t' = 0$, then Lemma 2.5.5 shows that $\mathcal{F}$ contains a $\{B\}$-free graph J that is induced in a realization of $\varepsilon' = (2s' - 1)^1 3^1 2^{2s'-1}$. A maximum induced matching in J has at size at most $s' - 1 < s < a$, so J is $\{A, C\}$-free, a contradiction. If $t' \neq 0$, by Lemma 2.5.4 $\mathcal{F}$ contains a $\{B\}$-free graph J' that is induced in a graph obtained by subdividing an edge of a realization of $K_1 \vee (s'K_2 + (t' - 1)K_1)$. Again J' is

$\{A, C\}$-free, another contradiction.

If $\Delta(C) = 2$, then by Lemma 2.5.5 the graph C is $P_3 + sK_2 + tK_1$ or $2P_3 + sK_2 + tK_1$ or $P_4 + sK_2 + tK_1$ for suitably bounded values of s, t.

If $C = P_3 + sK_2 + tK_1$ then $s \leq a - 2$ and $s + t \leq a - 1$. Consider the pair (d, e), where $d = 3^1 2^{3s+2t+1} 1^1$ and $e = 2^{3s+2t+3}$. Since (d, e) is not a counterexample pair, $\mathcal{F}$ must contain an induced subgraph of one of the realizations of e, since d has a realization inducing C, namely the graph obtained by adding edge $v_1 v_4$ to the path $v_1 v_2 \cdots, v_{3s+2t+3}$. Every realization of e is $\{A\}$-free, since otherwise $3(s + t + 1) \leq 3a \leq 3s + 2t + 3$, which is a contradiction. Realizations of e are also $\{B\}$-free since $\Delta(B) \geq 3$. Finally, deleting $s + t$ vertices from a realization of e leaves at most $s + t - 1$ components, so the realization is also $\{C\}$-free, a contradiction.

If $C = 2P_3 + sK_2 + tK_1$, then $s \leq a - 3$ and $s + t \leq a - 3$. In arguments similar to those of the last paragraph, the set $\mathcal{F}$ must contain an induced subgraph of one of the realizations of $e = 2^{3s+2t+7}$, but all such realizations are $\mathcal{F}$-free, a contradiction.

Hence $C = P_4 + sK_2 + tK_1$ where $s \leq a - 2$ and $s + t \leq a - 2$. If both s and t are 0, then $C = P_4$ and $\mathcal{F}$ must have an induced subgraph of $P_3 + K_2$ (otherwise $(2211, 21111)$ is a counterexample pair). Since $\Delta(B) \geq 3$, the induced subgraph is A. By our previous assumptions on A we conclude that $A = 2K_2$, and by Lemma 2.5.3 we find $\mathcal{F} = \{2K_2, P_4, \text{diamond}\}$. Otherwise, $s + t \geq 1$. Thus $\mathcal{F}$ must contain an induced subgraph of one of the realizations of $e = 2^{3s+2t+4}$ and we arrive at a contradiction as before in the argument for the case $C = P_3 + sK_2 + tK_1$.

Subcase 2: $b \geq 1$**.**

Since A has at least three vertices, and A is not $K_2 + K_1$, assume that $a \geq 2$ if $b = 1$ and $a \geq 1$ if $b = 2$.

By Lemma 2.5.4, $\mathcal{F}$ must contain an induced subgraph of a graph obtained by subdividing an edge of $K_1 \vee (aK_2 + (b-1)K_1)$.

If B is induced in an edge-subdivided $K_1 \vee (aK_2 + (b-1)K_1)$, then $B = K_1 \vee (pK_2 + qK_1)$ for integers p, q such that $p \leq a$ and $p + q \leq a + b - 1$. By Lemma 2.5.4, $\mathcal{F}$ contains an induced subgraph H of a graph obtained by subdividing an edge of $K_1 \vee (pK_2 + (q-1)K_1)$. This subgraph of H must be C. Hence C is induced in both the disjoint union of cycles having at most $3a+2b-1$ vertices and a graph obtained by subdividing an edge of $K_1 \vee (pK_2 + (q-1)K_1)$ where $p \leq a$ and $p + q \leq a + b - 1$.

If $\Delta(C) \leq 1$, then $C = sK_2 + tK_1$ for some s, t such that $s \leq p$ and $s + t \leq p + q$. If $t = 0$ then C is induced in A, contrary to our assumption. A similar contradiction occurs if $s = 0$. We assume that $s, t \neq 0$ (and as before, that C is not $K_2 + K_1$). By Lemma 2.5.4, $\mathcal{F}$ contains an induced subgraph H' of a graph obtained by subdividing an edge of $K_1 \vee (sK_2 + (t-1)K_1)$, where $s \leq p \leq a$ and $s + t \leq p + q - 1 \leq a + b - 2$. However, A is not induced in any realization of S' and neither is B, a contradiction.

If $\Delta(C) = 2$, the graph C contains vertex u of maximum degree in H. Since C is not P_3, K_3, or C_4, we have $C = P_4$. Since $(2211, 21111)$ is not a counterexample pair, $A = 3K_1$ or $A = K_2 + 2K_1$. However, when A is $3K_1$ or $K_2 + 2K_1$ we have respectively $(43221, 42222)$ and $(43322, 33332)$ as counterexample pairs, another contradiction.

If B is not induced in an edge-subdivided $K_1 \vee (aK_2 + (b-1)K_1)$, then C must be, in addition to being induced in a disjoint union of cycles having at most $3a + 2b - 1$ vertices. We again arrive at a contradiction using exactly the same argument as above when C was induced in an edge-subdivided $K_1 \vee (pK_2 + (q - 1)K_1)$.

2.5.2 Case: No graph in $\mathcal{F}$ has a dominating vertex

Recall that $A = aK_2 + bK_1$, where $a, b \geq 0$, and that $\Delta(C) \leq 2$. By Corollary 2.3.3, since B has no dominating vertex, it is $(|V(B)| - 2)$-regular and $|B|$ is even. If $|V(B)| = 4$ then B is C_4, contrary to a previous assumption, so assume that $|V(B)| \geq 6$ and hence $\delta(B) \geq 4$.

Since no graph in $\mathcal{F}$ has a dominating vertex, Theorem 2.3.1 implies that $\overline{\mathcal{F}} = \{\overline{A}, \overline{B}, \overline{C}\}$ is dominance monotone. If $b \geq 1$, then $\overline{A}$ has a dominating vertex, so the set $\overline{\mathcal{F}}$ was found in the previous subsection. Assuming that $\mathcal{F}$ contains no dominance monotone singleton or pair, we conclude that $\overline{\mathcal{F}}$ is equal to $\{2K_2, P_4, \text{diamond}\}$ and hence $\mathcal{F} = \{K_2 + 2K_1, C_4, P_4\}$. Suppose henceforth that $b = 0$, i.e., that $A = aK_2$ for some $a \geq 2$.

By Lemma 2.5.5, $\mathcal{F}$ contains an induced subgraph of at least one of the realizations of $\varepsilon = (2a-1)^1 3^1 2^{2a-1}$, and this induced subgraph is not A. Since $\delta(B) \geq 4$, neither is B induced in a realization of ε, and hence C must be. We proceed by considering the cases $\Delta(C) \leq 1$ and $\Delta(C) = 2$.

The statement $\Delta(C) \leq 1$ implies that $C = sK_2 + tK_1$, where $s \leq a - 1$ (with equality only if $t = 0$) and $s + t \leq a + 1$ by Lemma 2.5.5. Since we assumed that C is not induced in A, we have $t \neq 0$. Then Theorem 2.3.1 implies that $\overline{\mathcal{F}}$ is dominance monotone, and $\overline{\mathcal{F}}$ contains a graph with a dominating vertex. Thus the set $\overline{\mathcal{F}}$ was found in the previous subsection, where it was shown to be $\{2K_2, P_4, \text{diamond}\}$; however, this is a contradiction, since $\mathcal{F}$ was assumed to have two graphs with maximum degree at most 1.

If $\Delta(C) = 2$, then by Lemma 2.5.5 we have C is $P_3 + sK_2 + tK_1$ or $2P_3 + sK_2 + tK_1$ or $P_4 + sK_2 + tK_1$ for suitably bounded s and t. We may handle these cases using arguments very similar to those at the end of Subsection 2.5.1, noting that though B does not have a dominating vertex, its degrees are high enough for the arguments to work the same way.

2.6 Comments and questions

All of the dominance monotone sets mentioned in Section 2.1 are forbidden subgraph sets for subclasses of the split graphs. The triples $\{2K_2, P_4, \text{diamond}\}$ and $\{P_4, C_4, K_2 + 2K_1\}$ from Theorem 2.5.1, which respectively do allow C_4 or $2K_2$, show that families obtained from forbiddding a dominance monotone set can contain non-split graphs.

We have characterized the dominance monotone sets of size at most 3. Larger dominance monotone sets are also possible; in fact, there are infinitely many and arbitrarily large such sets.

Theorem 2.6.1 *Let $t \geq 1$. If $\mathcal{F}_t$ is the set of all graphs of order t, and $\mathcal{F}'_t$ is the set of all graphs with exactly t edges, then $\mathcal{F}_t$ and $\mathcal{F}'_t$ are dominance monotone.*

Proof. Take $t \geq 1$. Assume that $\mathcal{F}_t$ is the set of all graphs of order t. Let $d = (d_1, \cdots, d_n)$, $e = (e_1, \cdots, e_p)$ be two degree sequences such that $d \succeq e$ (terms of d and e are assumed to be positive integers). Assume further that e is forcibly $\mathcal{F}_t$-free; that is no realization of e contains an induced subgraph of order t. This implies that $p < t$. From Muirhead's Lemma, we have $n \leq p < t$; thus d must also be forcibly $\mathcal{F}_t$-free. Since d and e were arbitrary, we have our desired result for $\mathcal{F}_t$.

Likewise, if d and e are as above and e is forcibly $\mathcal{F}'_t$-free, then realizations of e have fewer than t edges, so the sum of the terms of e is less than $2t$ by the Handshaking Lemma. Since $d \succeq e$, the sum of terms in d equals the same number, and so every realization of d is $\mathcal{F}'_t$-free as well, establishing our result for $\mathcal{F}'_t$. $\square$

Observe that all known dominance monotone sets $\mathcal{F}$ have the property that $\overline{\mathcal{F}}$ is dominance monotone, even when $\mathcal{F}$ contains a dominating vertex, so we conjecture that the condition in Theorem 2.3.1 is not necessary: the complements of graphs in any dominance monotone set form a dominance monotone set. The difficulty in proving this lies in the dominance order's degree sequences not containing any 0 terms; it seems difficult to modify the poset to allow 0 terms without undesirable consequences.

2.7 Addendum

In this section, we take a closer look at the forcibly $\{P_4, C_4, K_2 + 2K_1\}$-free, $\{\text{diamond}, P_4, 2K_2\}$-free sequences as well as their realizations.

In [6], Merris describes the *threshold covered partitions* as the non-threshold partitions just under the threshold partitions in $\mathcal{D}_{2m}$. Merris gave this formal definition of non-threshold covered partitions: given u a threshold partition, v is a non-threshold covered partition if $u \succeq v$ and $F(v)$ can be obtained from $F(u)$ by moving a single box down from row i to row j $(j > i)$ where $u_i \geq u_j + 2$ and either $j = i + 1$ or $u_i - 1 = u_i + 1 = \cdots = u_{j-1} = u_j + 1$. By [8], the threshold covered partitions have graphical representation(s). Moreover, in [6], it has been shown that they are unigraphic.

In Proposition 2.5.2 we show that the the forcibly $\{P_4, C_4, K_2 + 2K_1\}$-free degree sequences have the following degree sequence characterization: $e_1 = (k+3)^k(k+1)^4$ covered by $(k+3)^k(k+2)^1(k+1)^2k^1$. It follows therefore that e_1 is a threshold covered graphic partition. Similarly, the complement of a $\{P_4, C_4, K_2 + 2K_1\}$-free graph, which is a $\{\text{diamond}, P_4, 2K_2\}$-free graph, is also a threshold covered graph. The degree sequences of these graphs have the following characterization: $e_2 = 2^4 0^k$.

Therefore, both the forcibly $\{P_4, C_4, K_2 + 2K_1\}$-free, and $\{\text{diamond}, P_4, 2K_2\}$-free sequences represent sub-classes of the class of threshold covered sequences. Now it is left to find the entire class, which leads to our next immediate project: to characterize the threshold covered graphs as well as the split covered graphs.

We end with this conjecture that we intend to take care of in our next project:

Conjecture 2.7.1 *Any threshold covered graph can be constructed starting with either a $2K_2$ or a C_4 and sequentially adding a finite number of either isolated or dominating vertices.*

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CHAPTER 3

The NGD-Circulant Graphs, $C_n(1, k)$

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Abstract

Nordhaus and Gaddum showed, for any given graph G, that $\chi(G) + \chi(\overline{G}) \leq n+1$, where χ denotes the chromatic number and n the number of vertices. Collins and Trenk established an analogous result for the distinguishing chromatic number. They proved, for any graph G, that $\chi_D(G) + \chi_D(\overline{G}) \leq n + D(G)$, where $D(G)$ is the distinguishing number of the graph. They called the class of graphs that satisfy equality in this bound NGD-graphs after Nordhaus and Gaddum. In this paper, we investigate the distinguishing chromatic number for the complements of circulant graphs $G = C_n(1,k)$. Some of our motivation comes from a similar work done by Barrus, Guillaume and Lantz for the circulant graphs $G = C_n(1,k)$. Naturally, we proceed to characterize the NGD-graphs for this particular class of graphs and consequently improve the bound of Collins and Trenk for this family. Lastly, we extend our investigation of the distinguishing chromatic number to a larger class of circulant graphs $G = C_n(1,S)$, where $S \subseteq \{1, \cdots, \lfloor \frac{n}{2} \rfloor\}$.

3.1 Introduction

All graphs in this paper are finite and simple. In 1996, motivated by Frank Rubin's problem in [6] regarding a blind professor who seeks to determine the minimum number of handle shapes (labels) needed to distinguish all his keys on a circular key ring, Albertson and Collins in [1] defined a new parameter known as the *distinguishing number* of a graph G, denoted $D(G)$. They defined a *distinguishing labeling* of a graph G to be a labeling which is not preserved by any non-identity automorphism and used $D(G)$ to denote the minimum number of labels needed to produce a distinguishing labeling. In other words, the labeling distinguishes the vertices from each other by destroying all the nontrivial symmetries of G. Some years later, Collins and Trenk in [4] tightened the conditions: they required that the labeling be proper, i.e., no two adjacent vertices may have the same color. The proper-coloring analogue of the distinguishing number of Albertson and Collins is the *distinguishing chromatic number*, $\chi_D(G)$. The following are some of the many interesting results published by Albertson, Collins and Trenk in [1] and [4]:

Graph G	$D(G)$	$\chi_D(G)$
C_3	3	3
C_4	3	4
C_5	3	3
C_6	2	4
C_{2n}, $n \geq 4$	2	3
C_{2n+1}, $n \geq 3$	2	3

Here C_n is the cycle graph on n vertices. More recently, motivated by these results and others, Barrus, Guillaume and Lantz took a close look in [2] at the structure of the circulant graphs $G = C_n(1, k)$ and used this knowledge to study their distinguishing chromatic number. Their main results determined the exact values of $\chi_D(G)$ for $G = C_n(1, 2)$ and $G = C_{2k}(1, k \pm 1)$. They showed that $\chi_D(C_n(1, k)) \leq 4$ and conjectured that it is 3 with few exceptions.

In 2013, inspired by the classic Nordhaus and Gaddum theorem in [5], which stated that for any graph G,

$$\begin{aligned} 2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \text{ and} \\ n \leq \chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2, \end{aligned} \tag{2}$$

Collins and Trenk in [4] gave these analogues for the distinguishing chromatic number:

$$\begin{aligned} 2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) \leq n + D(G); \\ n \leq \chi_D(G) \cdot \chi_D(\overline{G}) \leq \left(\frac{n+D(G)}{2}\right)^2. \end{aligned} \tag{3}$$

They called the graphs that satisfy the upper bound in (2) with equality *NG-graphs* and those for which equality holds on the right in (3) *NGD-graphs*. As one might expect from a generalization, the upper bounds in (3) are not always sharp. For example, for $G = C_5$, since G is its own complement, we have $\chi_D(G) + \chi_D(\overline{G}) = 3 + 3 < n + D(G) = 5 + 3$.

These facts prompt us to take on a natural extension of the results in [2] wherein we will investigate the distinguishing chromatic number of the complements of $C_n(1, k)$ and characterize all the NGD-graphs of the form $C_n(1, k)$. Moreover, we provide an improvement for the upper bounds in (3) for these graphs.

3.2 Preliminaries

In this section, we recall basic terminology and notions for the distinguishing chromatic number of a graph and related concepts, and we state some facts and proven results that will be useful to us later.

In general, for $n \in \mathbb{N}$ and $n \geq 3$, the circulant graph $G = C_n(1, k)$, where $1 \leq k \leq n - 1$, is the simple graph with vertex set $\{v_i : i \in \mathbb{Z}_n\}$(where $\mathbb{Z}_n$ is the set of the integers modulo n) such that two vertices v_i and v_j are adjacent if and only if for s equal 1 or k we have either $i + s = j \bmod n$ or $j + s = i \bmod n$. If we call the edge $v_i v_j$ such that $i + s = j \bmod n$ or $j + s = i \bmod n$ type-s edge, then we see that $C_n(1, k)$ consists simply of type-1 and type-k edges. Observe that $C_n(1, k) = C_n(1, n - k)$ and the cycle graphs C_n are circulant graphs $C_n(1, k)$ where $k = 1$. Hence, for the remaining of the paper, we assume that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$

.

Under this assumption, the circulant graphs $C_n(1, k)$ fall naturally into three classes: the cycle graphs where $k = 1$, the 3-regular graphs $C_{2k}(1, k)$, and the tetravalent graphs. However, in this paper, we partition the circulant graphs $C_n(1, k)$ into two useful categories (for reasons that will become apparent later): the *triangle-free* circulant graphs and the *triangle-inducing* circulant graphs. The triangle-free circulant graphs are the circulant graphs $C_n(1, k)$ that contain no K_3

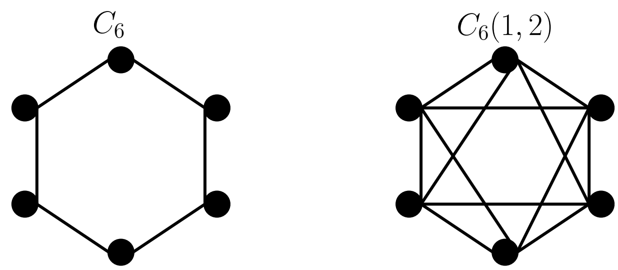

Figure 10. The cycle graph C_6 and the circulant graph $C_6(1, 2)$.

(a triangle) as an induced subgraph. The triangle-inducing circulant graphs are the graphs $C_n(1, k)$ that are not triangle-free. It is worth observing that if a graph G contains an induced triangle, then the size of the independent set of vertices in its complement $\overline{G}$ is at least 3. At the end of the section, we shall characterize all the triangle-free graphs $C_n(1, k)$.

The distinguishing number $D(G)$ of a graph G is the least integer r such that G has a vertex labeling with r labels that is not preserved under any non-trivial automorphism. In the case of a proper labeling, r is called the distinguishing chromatic number $\chi_D(G)$ of G, which often differs from the chromatic number $\chi(G)$ of G. In the case of the chromatic number, the only requirement is that the labeling is proper. Thus, for any graph G, we have $\chi(G) \leq \chi_D(G)$.

For the remaining part of the section, we characterize the triangle-free graphs $C_n(1, k)$ and recall some previously known results that will be used throughout this paper. First, we describe all the circulant graphs $G = C_n(1, k)$ that contain an induced triangle.

Theorem 3.2.1 *A circulant graph $G = C_n(1, k)$ is triangle-free unless G is one of the following graphs:*

- C_3;
- $C_n(1, 2)$ *for* $n \geq 4$;

- $C_n(1, \lfloor \frac{n}{2} \rfloor) = C_n(1, \frac{n-1}{2})$ *for odd* $n \geq 5$;

- $C_{3k}(1, k)$ *for* $k \geq 2$.

Proof.

Let $G = C_n(1, k)$ be a circulant graph and denote the vertices of G consecutively as $v_1, \cdots, v_n$. By definition of circulant graphs, there exists an edge $v_i v_j$ if and only if $i + s = j \bmod n$ or $j + s = i \bmod n$, where $s \in \{1, k\}$. Now suppose G has an induced triangle generated by vertices v_i, v_j, v_l. If all three edges of the triangle are type-1 edges, then $G = K_3$. If exactly two edges of the triangle are type-1 edges, then, the other edge must be a type-2 edge; for otherwise we have $s > \lfloor \frac{n}{2} \rfloor$, which is contrary to our assumption on k. If only one edge of the triangle is a type-1 edge, then, $n = 2k + 1$. By solving for k, we see that $G = C_n(1, \lfloor \frac{n}{2} \rfloor)$. If all three edges of the triangle are type-k edges, then $G = C_{3k}(1, k)$. Hence, we have our desired result.

$\square$

Remark 3.2.2 *A similar analysis as in the proof of Theorem 3.2.1 shows that if we suppose $G = C_n(1, k)$ contains a complete graph on four vertices, then $k = 2$. From there, it is easy to see that for $n > 5$, no graph $G = C_n(1, k)$ contains an induced K_4(a complete graph on four vertices). Therefore, the only K_4-inducing graphs $G = C_n(1, k)$ are $C_4(1, 2)$ and $C_5(1, 2)$, which are complete graphs.*

Here are some more relevant results: The first two are well known in graph theory.

Proposition 3.2.3 ([7]) *For any graph G, $\frac{n(G)}{\alpha(G)} \leq \chi(G)$.*

Proposition 3.2.4 *For any graph G, $\omega(G) \leq \chi(G) \leq \chi_D(G)$, where $\omega(G)$ is the clique number of G (size of the largest clique in G).*

Proposition 3.2.5 (Barrus-Guillaume-Lantz [2]) *Let $G = C_n(1,k)$, $n \neq k^2 \pm 1$ and $G \neq C_{2k}(1,k-1)$ be a tetravalent graph; then Aut(G)$= D_{2n}$, where D_{2n} is the dihedral group.*

Proposition 3.2.6 (Collins and Trenk [4]) *If G is a complete graph, then $D(G) = \chi_D(G) = \chi(G) = n$, and $D(\overline{G}) = \chi_D(\overline{G}) = n$ while $\chi(\overline{G}) = 1$.*

Remark 3.2.7 *Let $G = C_n(1,k)$ and $S = \{1,k\}$. Denote the vertices of G consecutively as $v_1, v_2, \cdots, v_n$. Then, by definitions of a circulant graph and complement of a graph, if $i + s = j$ mod n or $j + s = i$ mod n, where $s \in S$, then v_i, v_j are not adjacent in $\overline{G}$.*

Remark 3.2.8 *The automorphism group of a graph is the same as the automorphism group of its complement. Thus, for any graph G, $D(G) = D(\overline{G})$.*

Lemma 3.2.9 *Let $G = C_n(1,k)$. Then $D(G) > 1$.*

Proof. Suppose $D(G) = 1$. Then we have all the nontrivial elements of D_{2n}, namely the rotations and the reflections (all the *rigid* symmetries), as non-trivial label preserving automorphisms, a contradiction. $\square$

The last theorem suggests the following layout for the rest of the paper: in Sections 3 and 4, we investigate these special (triangle-inducing) graphs given in Theorem 3.2.1. In Section 5, we direct our attention to the triangle-free graphs. The last section contains an extension of the results obtained in Section 5 to a larger class of circulant graphs. Also, for the remainder of this paper that we denote the vertices of $G = C_n(1,k)$ consecutively as $v_1, \cdots, v_n$.

3.3 $G = C_n(1,2)$

In this section, we calculate $\chi_D(\overline{G})$ and characterize all the NGD-graphs of the form $C_n(1,2)$. Our main approach is to find a lower bound on the chromatic

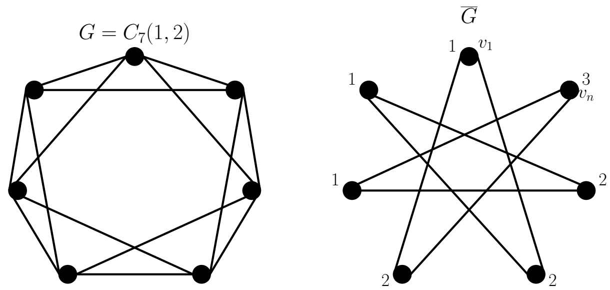

Figure 11. An example of $C_n(1,k)$, $n > 6$ and $n \equiv 1 \bmod 3$

number $\chi(\overline{G})$ using Proposition 3.2.3. We proceed to construct a proper labeling σ using a minimum number of labels starting from the prescribed lower bound. Ultimately, we show that this proper labeling is distinguishing by using Proposition 3.2.5.

For $n < 6$, G is a complete graph. By Remark 3.2.2 for $n \geq 6$, G contains no K_4 subgraph. Thus, by Theorem 3.2.1, $\omega(G) = 3$, which leads to $\alpha(\overline{G}) = 3$. Thus, by Proposition 3.2.3, $\chi(\overline{G}) \geq \lceil \frac{n(G)}{3} \rceil$. In fact, the chromatic number of the complement of G is exactly $\lceil \frac{n(G)}{3} \rceil$, and we establish this by constructing a proper labeling σ of $\overline{G}$. We proceed to confirm that σ is distinguishing.

Proposition 3.3.1 *For $G = C_n(1,2)$, we have $\chi(\overline{G}) = \lceil \frac{n}{3} \rceil$.*

Proof. We construct a proper labeling σ of $V(\overline{G})$ using $\lceil \frac{n}{3} \rceil$ labels. Let $C = \{1, \cdots, \lceil \frac{n}{3} \rceil\}$ be the set of labels, and define the labeling function σ as

$$\begin{aligned} \sigma: \quad & V(G) \to C \\ : \quad & v_i \to \lceil \frac{i}{3} \rceil \end{aligned}$$

for $1 \leq i \leq n$. By Remark 3.2.7, σ is a proper labeling of $V(\overline{G})$. Hence, our desired result.

□

Next we show that the labeling σ above is a proper distinguishing labeling.

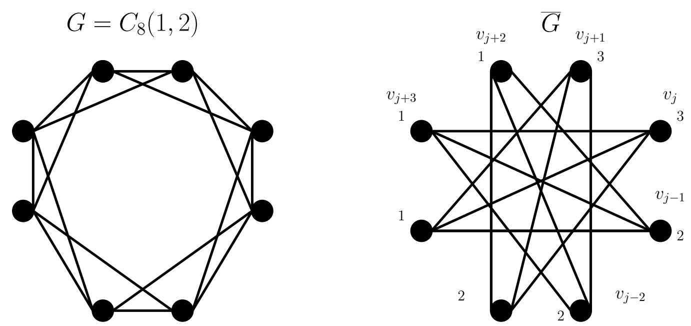

Figure 12. An example of $C_n(1,k)$, $n > 6$ and $n \equiv 2 \bmod 3$

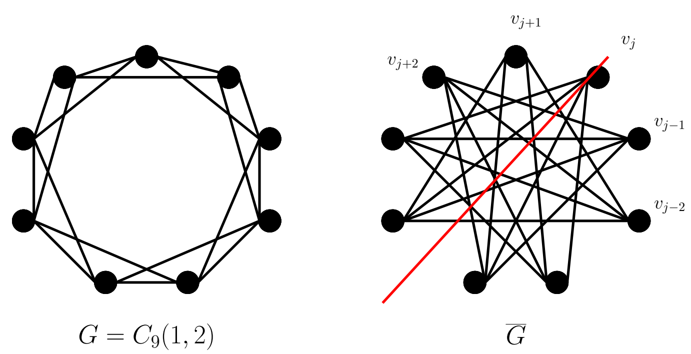

Figure 13. An example of $C_n(1,k)$, $n > 6$ and $n \equiv 0 \bmod 3$

Proposition 3.3.2 *Let $G = C_n(1,2)$, where $n > 6$. Then $\chi_D(\overline{G}) = \lceil \frac{n}{3} \rceil$.*

Proof.

For $n > 6$, we show that the proper labeling σ is distinguishing. By Proposition 3.2.5, $\mathrm{Aut}(G) \cong D_{2n}$. Next, we consider the following three cases:

Case 1. $n \bmod 3 = 1$.

In this case, the label $\lceil \frac{n}{3} \rceil$ under σ is assigned to exactly one vertex, which is v_n by our construction. Thus, v_n cannot be permuted with any other vertex under a label-preserving automorphism; that is, v_n is fixed under any non-trivial automorphism, which by Proposition 3.2.5 belongs to D_{2n}(see Figure 11). This eliminates the possibility of a nontrivial rotation as a symmetry. It suffices to show there can't be any reflection about an axis containing v_n. Since any such reflection would have to permute v_{n-1} with v_1, which are labeled differently with respect to σ (see Figure 11). Therefore, σ is distinguishing.

Case 2. $n \bmod 3 = 2$.

In this case, the label $\lceil \frac{n}{3} \rceil$ under σ is assigned to exactly one pair of consecutive vertices which are v_n, v_{n-1}. Any nontrivial label-preserving rotation or reflection would have to permute this pair. Note that such nontrivial rotation would permute v_{n-2} with v_{n-1} which cannot happen since they have different labels. Such nontrivial reflection would take v_{n-2} to v_1, which cannot happen since they have different labels (see Figure 12).

Case 3. $n \bmod 3 = 0$.

Every label is used exactly 3 times. No nontrivial label-preserving rotation will preserve all labels. It remains to show that there exists no nontrivial label-preserving reflection. To show this, let's suppose that Ω is such a reflection about the axis of symmetry x. Either x contains a vertex v_j for some j or not, where $1 \leq j \leq n$. If it does contain v_j, then under Ω, v_{j+2} is permuted with v_{j-2}. Since each label is only used exactly 3 times and $n > 6$, vertices v_{j+2} and v_{j-2} have different labels under σ. Thus, such reflection does not exist. If x does not contain any vertex and instead goes between v_j and v_{j+1}, then under Ω, v_j is permuted with v_{j+1} and v_{j-1} is permuted with v_{j+2}, which for similar reasons carry different labels. Again, such reflection does not exist. Since v_j was arbitrary, σ is distinguishing.

Hence, by Cases 1, 2, 3, we have our desired result.

□

We are ready to characterize the NGD-graphs for $C_n(1,2)$. To this aim, we use a theorem from [2] that was proven there using the concept of metric dimension. Here, we give a simpler proof which only uses Proposition 3.2.5 and a construction.

Theorem 3.3.3 *Let $G = C_n(1,2)$. For $n \leq 6$, $\chi_D(G) = n$; for $n \geq 7$, $\chi_D(G) = 4$.*

Proof. For $n \leq 5$, $\chi_D(G) = n$ since G is a complete graph. By construction(it is a good exercise; note that the induced triangles limit our labeling options), it can be verified that when $n = 6, 7, 8$, $\chi_D(G)$ is 6, 4, and 4 respectively. For $n \geq 9$, we give the following construction which shows $\chi_D(G) = 4$. Note that the given construction does not work for $7 \leq n \leq 8$.

Let $n \geq 9$ and $G = C_n(1,2)$. From Proposition 3.2.5 we have $Aut(G) = D_{2n}$ for $n \geq 6$. We assign the labels 1,2,3,1 to vertices v_1, v_2, v_3, v_4 respectively. Next, starting at v_5, we assign the ordered set of labels $(2, 3, 4)$ repeatedly to vertices $v_5, \cdots, v_{n-3}$ consecutively. That is, vertices $v_5, v_6, v_7, v_8, v_9, \cdots$ are labeled $2, 3, 4, 2, 3, 4, \cdots$ and so on. To complete our labeling, we consider the following three cases:

Case 1: If v_{n-3} is labeled 4 , then assign 2, 3, 4 to v_{n-2}, v_{n-1}, v_n respectively.

Case 2: If v_{n-3} is labeled 3, then assign 4, 2, 3 to v_{n-2}, v_{n-1}, v_n respectively.

Case 3: If v_{n-3} is labeled 2, then assign 1, 3, 4 to v_{n-2}, v_{n-1}, v_n respectively.

We remark that a "cushion" of at least 2 vertices is needed between v_4 and v_{n-2} for this construction to work . Thus, we need n to be at least 9.

The labeling as presently constructed is proper, which is easily verified by inspection. It remains to show that it is distinguishing. To this end, realize that the vertex v_3 is fixed since it is the only vertex labeled 3 that is not adjacent to a vertex labeled 4. Thus, no nontrivial label-preserving rotation is possible. Moreover, since v_4, v_2 are labeled differently, no nontrivial label-preserving reflection is possible. Hence, the labeling is distinguishing.

When n is 7 and 8, Figure 14 shows a proper distinguishing labeling, which is verifiable by inspection

So far, for $n \geq 9$ we have shown that $\chi_D(G) \leq 4$. Since we cannot have a proper label a triangle using less than three labels, easy to see $\chi_D(G) \geq 3$. Furthermore, with 3 available colors, observe that the proper labeling requirement

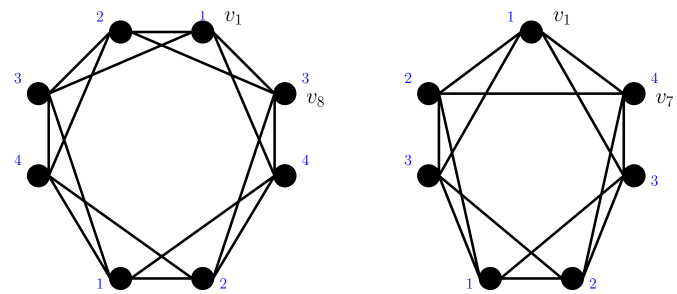

Figure 14. Distinguishing labeling of $C_8(1,2), C_7(1,2)$

dictates a specific pattern, which is the repetition of an ordered arrangement of all 3 labels. However, such a proper labeling is only possible in the case where n mod $3 = 0$, which yields a nontrivial label-preserving symmetry. Thus, $\chi_D(G) = 4$.

□

Theorem 3.3.4 *Let $G = C_n(1,2)$. Then G is not an NGD- graph if and only if $n \geq 7$. Moreover, $\chi_D(G) + \chi_D(\overline{G}) = 4 + \lceil \frac{n}{3} \rceil$.*

Proof.

Clearly, $D(G) > 1$. Otherwise, we have all rigid non trivial symmetries. Thus, for $n \geq 7$, by Theorem 3.3.3 and Proposition 3.3.2, we have that $\chi_D G + \chi_D(\overline{G}) = 4 + \lceil \frac{n}{3} \rceil < n + D(G)$. Therefore, G is not a NGD-graph by definition.

For $n < 6$, G is a complete graph. Thus, by [4] $D(K_n) = n$ and $\chi_D G + \chi_D(\overline{G}) = n + D(G) = 2n$. For $n = 6$, $\overline{G}$ is a $3K_2$ or a matching. We claim that $D(\overline{G}) = 3$. It is not hard to see that the claim holds true: first, observe that the labeling in Figure 15 is distinguishing; now, it suffices to show that we cannot achieve a distinguishing labeling using less than 3 labels. Clearly, we cannot do with one label. Suppose we use two labels. Since $\overline{G} = 3K_2$, thus adjacent pair of vertices cannot have the same label; otherwise, this pair is not distinguishable. Therefore, boths labels must be used on each independent edge, which makes the edges not distinguishable. Thus, we cannot achieve a distinguishing labeling with two labels.

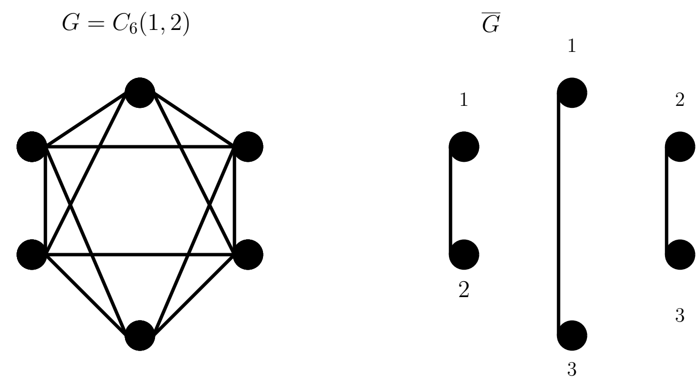

Figure 15. $\chi_D(\overline{G}) = 3$

However, it can be done with 3 labels according to Figure 15. Thus $D(\overline{G}) = \chi_D(\overline{G}) = 3$ as stated in claim. Therefore, by Theorem 3.3.3 and Remark 3.2.8, we have for $n = 6$ the desired result $\chi_D(G) + \chi_D(\overline{G}) = 6 + 3 = 9 = n + D(G)$. □

Theorem 3.3.5 *Let $G = C_n(1,2)$, for $n \geq 7$. Then,*

$$\begin{aligned} 2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) = 4 + \left\lceil \frac{n}{3} \right\rceil; \\ n \leq \chi_D(G) \cdot \chi_D(\overline{G}) = 4 \cdot \left\lceil \frac{n}{3} \right\rceil. \end{aligned} \tag{4}$$

Proof.

By Remark 3.2.4, the two lower bounds which are the ones stated in [4] remain unchanged. The upper bounds follow from all propositions and theorems in this section.

□

3.4 $C_n(1, \lfloor \frac{n}{2} \rfloor)$ and $C_{3k}(1,k)$

In each case, we investigate the distinguishing chromatic number of both G and $\overline{G}$, and characterize the NGD graphs.

3.4.1 $G = C_{3k}(1,k)$

In this case, we consruct two separate labelings, one for G and the other for $\overline{G}$ which show that $\chi_D(G) \leq 4$ and $\chi_D(\overline{G}) \leq k + 1$.

Proposition 3.4.1 *Let $G = C_{3k}(1, k)$ where $k > 2$. Then $\chi_D(G) \leq 4$.*

Proof. Assume $G = C_{3k}(1, k)$. Apply the labeling described below to G.

To enhance visualization, consider the following $3 \times k$ table where $B_1 :=$ row 1, B_2 :=row 2, $B_3 :=$ row 3, where row i represents this (carefully chosen so each column has distinct numbers) ordered arrangement of the set $S = \{1, 2, 3\}$ of labels.

B_1	1	2	3	2	3	$\cdots$	1
B_2	2	3	1	3	1	$\cdots$	2
B_3	3	1	2	1	2	$\cdots$	3

We now proceed with this algorithm:

(1)Assign label 4 to v_n.

(2) From v_1 to v_{n-1} alternate B_1, B_2, B_3 starting with B_1; that is to assign the arranged labels in row 1 to $v_1, \cdots, v_k$ consecutively, those of B_2 to $v_{k+1}, \cdots, v_{2k}$, and so on. Note that last label of row B_3 is not used since v_n is labeled 4.

Clearly from the table and inspection of Figure 16, we see that for arbitrary j, such that $1 \leq j \leq n$, each of the pairs of vertices $\{v_j, v_{j+1}\}, \{v_j, v_{j-1}\}, \{v_j, v_{j+k}\}$ has two distinct labels. Since j is fixed, we conclude that the labeling is a proper one. Next we show that it is distinguishing: clearly v_n is fixed since it is the only vertex labeled 4. Thus, by Proposition 3.2.5, there is no nontrivial label-preserving rotation as a symmetry. Moreover, since for v_k, v_{2k} have different labels, we thus have no nontrivial label-preserving reflection as a symmetry. Therefore, we have a proper distinguishing labeling. □

Next, we investigate the distinguishing chromatic number of the complements of $G = C_{3k}(1, k)$. By Proposition 3.2.3, $k \leq \chi(\overline{G})$. We shall show that $k \leq \chi(\overline{G}) \leq \chi_D(\overline{G}) \leq k + 1$.

Proposition 3.4.2 *Let $G = C_{3k}(1, k)$, $k > 2$. Then $\chi_D(\overline{G}) \leq k + 1$.*

$C_9(1,3)$

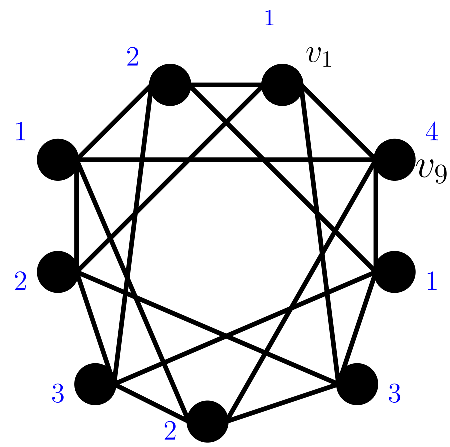

Figure 16. A proper distinguishing labeling of $C_9(1,3)$

Proof. Assume $G = C_{3k}(1,k)$, for $k > 2$. By Proposition 3.2.3, $k \leq \chi_D(\overline{G})$. We shall show that $\chi_D(\overline{G}) \leq k+1$ and leave the equality as a conjecture. To this aim, let's apply the labeling described below to the vertices of G.

We provide a table to enhance visualization: consider the following $1 \times k$ table where $B_1 :=$ row 1, where B_1 represents an ordered arrangement of the set $S = \{1, 2, \cdots, k\}$ of labels.

B_1	1	2	3	4	5	$\cdots$	k

We now proceed with this algorithm:

(1)Assign label $(k+1)$ to v_n.

(2) Superpose B_1 starting at v_1 to v_k, then from v_{k+1} to v_{2k}, and so on. Note that the label assigned to v_n is not coming from B_1.

By Remark 3.2.7, it is not hard to see that this is a proper labeling of vertices of $\overline{G}$. It remains to check that the labeling is distinguishing. To this aim, first realize that v_n is fixed. Thus, by Proposition 3.2.5, there is no nontrivial label-preserving rotation as a symmetry. Moreover, since for v_1, v_{n-1} have different labels, we have no reflection as a nontrivial label-preserving symmetry. Therefore, we have a proper distinguishing labeling as shown in Figure 16.

□

Theorem 3.4.3 *For all $k > 2$, $G = C_{3k}(1, k)$ is a non-NGD graph and*

$$\begin{aligned} \chi_D(G) + \chi_D(\overline{G}) &\leq 5 + k; \\ \chi_D(G) \cdot \chi_D(\overline{G}) &\leq 4(k+1). \end{aligned} \tag{5}$$

Proof. Result holds by the two previous propositions. □

Before we close this case, it is worth highlighting one important observation that may help with the next conjecture.

Observation 3.4.4 *For $k > 3$, when $3k$ is the least common multiple (LCM) of 3 and k, then there exists at least 3 cliques of size k in $\overline{G}$.*

It is not hard to see that since $3k$ is the *LCM* of 3 and k, the vertices v_{j+3d}, where $0 \leq d \leq k-1$, form a clique.

Conjecture 3.4.5 *Let $G = C_{3k}(1, k)$, for $k \geq 3$. Then $\chi_D(\overline{G}) = k + 1$.*

It is not hard to check that this is the case for $C_9(1, 3)$.

3.4.2 $G = C_n(1, \lfloor \frac{n}{2} \rfloor)$, $n > 5$ and n is odd

Similarly as above, we consruct two separate labelings, one for G and the other for $\overline{G}$, which show that $\chi_D(G) \leq 4$ and $\chi_D(\overline{G}) \leq \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$.

Proposition 3.4.6 *Let $G = C_n(1, \lfloor \frac{n}{2} \rfloor)$, where n is odd and greater than 5. Then $\chi_D(G) \leq 4$.*

Proof. For odd n and $n > 5$, assume $G = C_n(1, k)$, $k = \lfloor \frac{n}{2} \rfloor$. Apply the labeling described below to G.

To enhance visualization, consider the following $2 \times k$ table where $B_1 :=$ row 1, B_2 :=row 2, where row i represents an ordered arrangement of the set $S = \{1, 2, 3\}$ of labels.

B_1	1	2	3	1	2	3	$\cdots$
B_2	2	3	1	2	3	1	$\cdots$

We now proceed with this algorithm:

(1)Assign label 4 to v_n.

(2) From v_1 to v_{n-1} alternate B_1, B_2 starting with B_1; that is, begin by assigning the labels in row 1 of the table to vertices $v_1, \cdots, v_k$ consecutively and similarly using labels of row 2 to vertices $v_{k+1}, \cdots, v_{2k}$.

Clearly from the table and by inspection, we see that for arbitrary j, such that $1 \leq j \leq n$, all the vertex v_j does not have the same with any of the vertices $v_{j+k}v_{j-k}, v_{j-1}, v_{j+1}$. Since j is arbitrary, we conclude the labeling is a proper one. Next we show that it is distinguishing: clearly v_n is fixed since it is the only vertex labeled 4. Thus, by Proposition 3.2.5, there is no nontrivial label-preserving rotation as a symmetry. Moreover, since for v_k, v_{-k} have different labels, we thus have no nontrivial label-preserving reflection as a symmetry. Therefore, we have a proper distinguishing labeling. Hence, the result of the proposition follows.

$\square$

Next, we investigate the distinguishing chromatic number of the complement of $G = C_n(1, k)$, odd n and $k = \lfloor \frac{n}{2} \rfloor$. We will use Remark 3.2.7 to show that $\chi_D(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor$. Thus, we have the following proposition.

Proposition 3.4.7 *Let $G = C_n(1, \lfloor \frac{n}{2} \rfloor)$, odd n. Then $\chi_D(\overline{G}) \leq \frac{n-1}{2}$.*

Proof. First we show that we can obtain a proper labeling of $\overline{G}$ using $k = \lfloor \frac{n}{2} \rfloor$ labels. To this aim, let $C = \{1, \cdots, k\}$ be the set of labels. Consider the following induced triangles generated in G by these subsets of vertices $\{v_1, v_{1+k}, v_{1-k}\}$,

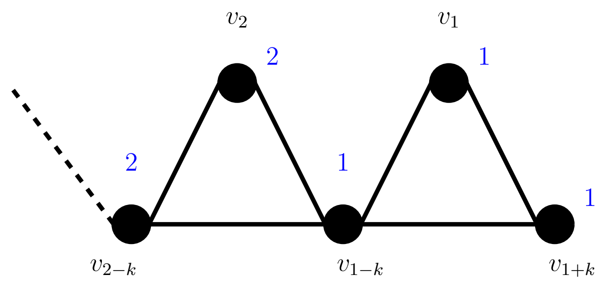

Figure 17. Induced triangles in G

$\{v_2, v_{2+k}, v_{2-k}\}, \cdots, \{v_k, v_{2k}, v_{k-k}\}$. This gives us a total of k triangles (see Figure 17). It is not hard to see that the set of vertices of these triangles is equal $V(G) = V(\overline{G})$. Call T_1 the triangle formed by vertices v_1, v_{1+k}, v_{1-k} , T_2 the triangle with vertices v_2, v_{2+k}, v_{2-k} and so on. Since $v_{j+k} = v_{j-1-k}$ for every j, thus T_1 intersects T_2 at exactly one vertex, T_2 intersects T_3 at exactly one vertex and so on. Assign label 1 to T_1, then label 2 to vertices of T_2 that are not yet labeled, label 3 to vertices of T_3 that are not yet labeled, and so on. By Remark 3.2.7, this labeling is proper on $\overline{G}$ (not on G).

Next we show that it is distinguishing. Observe label 1 is used three times, two of them on a consecutive pair of vertices. Suppose there exists a non-trivial label preserving rotation, at least one of the vertices v_{1+k}, v_{1-k} would be permuted with a vertex that is labeled other than 1. Similarly, for any nontrivial label-preserving reflection, at least one of the vertices v_1, v_{1+k}, v_{1-k} would be premuted to a vertex labeled other than 1. Thus, by Proposition 3.2.5, the labeling is distinguishing. Hence, the desired result.

□

Theorem 3.4.8 *Let $G = C_n(1, \lfloor \frac{n}{2} \rfloor)$, for odd n. Then G is a non-NGD graph if and only if n greater than 5. Moreover*

$$\chi_D(G) + \chi_D(\overline{G}) \leq 4 + \frac{n-1}{2};$$
$$\chi_D(G) \cdot \chi_D(\overline{G}) \leq 2(n-1). \tag{6}$$

Proof.

Necessity follows from Propositions 3.4.6, 3.4.7 and the fact that $D(G) > 1$.

Sufficiency follows from Propositions 3.4.6, 3.4.7 and Theorem 3.3.4.

□

3.5 The Triangle-free circulant graphs $C_n(1, k)$

Let $G = C_n(1, k)$ be a triangle-free circulant graph. We have by Proposition 3.2.3 that $\chi(\overline{G}) \geq \lceil \frac{n}{2} \rceil$. As in section 3, our main approach is to use this information to construct a proper labeling σ which shows that $\chi(\overline{G}) = \lceil \frac{n}{2} \rceil$. Moreover, we shall show that this proper labeling σ is distinguishing for the complements of all triangle-free graphs $G = C_n(1, k)$ such that $G \neq C_6(1, 3)$, or $C_8(1, 3)$, thus proving that $\chi(\overline{G}) = \chi_D(\overline{G}) = \lceil \frac{n}{2} \rceil$ for any triangle-free graph G other than $C_6(1, 3)$, and $C_8(1, 3)$. Lastly, we conclude that the only triangle-free NGD-graphs of the form $C_n(1, k)$ are $G = C_6(1, 3)$ and $G = C_8(1, 3)$.

First, let's introduce these important results from [2] that will be used later in this section:

Theorem 3.5.1 *If $G = C_{2k}(1, k-1) \cong C_{2k}(1, k+1)$, where $k \geq 5$. then $\chi_D(G) = 5$.*

Proof. See [2]. □

Theorem 3.5.2 *Let $G = C_n$. Then $\chi_D(C_n)$ is equal 4, 3, or 4 for $n = 4, 5,$ or 6 respectively, $\chi_D(C_{2n+1}) = 3$ for all $n \geq 3$, and $\chi_D(C_{2n}) = 3$ for all $n \geq 4$.*

Proof. See [3] or the table in Section 1. □

Proposition 3.5.3 *let $G = C_n(1,k)$ be a triangle-free circulant graph. Then* $\chi(\overline{G}) = \lceil \frac{n}{2} \rceil$.

Proof. As indicated above, we construct a labeling σ of $V(\overline{G})$ using $\lceil \frac{n}{2} \rceil$ labels. To this aim, let $C = \{1, \cdots, \lceil \frac{n}{2} \rceil\}$ be a set of labels and define the labeling function σ as follows:

$$\begin{aligned} \sigma: \quad & V(G) \to C \\ : \quad & v_i \to \lceil \frac{i}{2} \rceil \end{aligned}$$

for $1 \leq i \leq n$. By Remark 3.2.7, σ is a proper labeling of $V(\overline{G})$. Hence, our desired result follows from Proposition 3.2.3. $\square$

Next we show that the labeling σ above is distinguishing for all the complements of the triangle-free circulant graphs $G = C_n(1,k)$ such that $G \neq C_6(1,3)$, $C_8(1,3)$. Most importantly, we calculate the distinguishing chromatic number for all the complements of triangle-free graphs of the form $C_n(1,k)$.

Proposition 3.5.4 *Let $G = C_n(1,k)$ be a triangle-free graph such that $G \neq C_6(1,3)$, $C_8(1,3)$. Then σ above is distinguishing and $\chi(\overline{G}) = \chi_D(\overline{G}) = \lceil \frac{n}{2} \rceil$. If $G = C_6(1,3)$, or $C_8(1,3)$, then $\chi_D(\overline{G})$ is equal 4, or 5 respectively.*

Proof. We give a proof which consists of three cases: in case 1, we take care of all odd n; in case 2, all even n such that $G \neq C_6(1,3)$, $C_8(1,3)$; case 3 covers the two unique graphs $C_6(1,3)$, and $C_8(1,3)$.

1. Let $G = C_n(1,k)$, where n is odd, be a triangle-free graph. We show the proper labeling σ above is a distinguishing labeling of $\overline{G}$. To this aim, observe that the label $\lceil \frac{n}{2} \rceil$ is assigned to only one vertex which is v_n by construction. Thus v_n is fixed under all autmorphisms. Thus, no nontrivial label-preserving rotation as a symmetry can occur. Since v_{n-1}, v_1 are assigned different labels

under σ, no nontrivial label-preserving reflection can occur. It remains to check for non rigid symmetry. Observe that since v_n is fixed, that leads to v_{n-2}, v_{n-1} to be fixed since v_{n-2}, v_{n-1} have same labels but only v_{n-2} is adjacent to v_n. Since v_{n-2} is fixed, thus by similar reasoning, we have v_{n-4}, v_{n-3} being fixed also. Repeating this process, we end up with all vertices being fixed. Hence, σ is distinguishing and thus $\chi_D(\overline{G}) = \chi(\overline{G}) = \lceil \frac{n}{2} \rceil$.

2. Let $G = C_n(1,k)$, where n is even, be a triangle-free graph such that G is not $C_6(1,3)$ nor $C_8(1,3)$. We show the proper labeling σ above is distinguishing in $\overline{G}$.

    - Let $k \neq 3$. Note that each label is only used twice and v_1, and v_2 have same labels, namely label 1. Furthermore, v_2 is adjacent to both vertices labeled $\frac{n}{2}$ where as v_1 is only adjacent to one of them. Thus, v_1, v_2 are fixed under any non trivial automorphism. A similar argument can be made for any pair of vertices that share same label: suppose that for a fixed j, the vertices v_j, v_{j+1} share the same label. Thus, by Remark 3.2.7, v_{j+1} in $\overline{G}$ is adjacent to both v_{j-1}, v_{j-2} which are assigned the same label under σ while v_j is only adjacent to one of them. Hence, v_{j+1}, v_j are fixed under any nontrivial automorphism. Hence, we have our desired result.

    - Let $k = 3$. Consider σ as described in Proposition 3.5.3. Suppose that for the same j, the vertices v_j, v_{j+1} share same label. Thus, by Remark 3.2.7, v_j in $\overline{G}$ is adjacent to both v_{j+4}, and v_{j+5} which are assigned the same label under σ while v_{j+1} is only adjacent to one of them. Hence, v_{j+1}, and v_j are fixed under any nontrivial automorphism.

3. Let $G = C_6(1,3)$, or $C_8(1,3)$. Observe that for $n = 6, 8$, $\overline{G}$ is simply the union

of two independent cliques of equal size. Thus, by Proposition 3.2.6, we need a minimum of $\frac{n}{2}$ labels for each clique. If the same labels are reused on both cliques, then there exists a non trivial label preserving automorphism; namely the one that permutes the vertices with matching labels. Thus, $\chi_D(\overline{G}) = \frac{n}{2} + 1$.

$\square$

Now we are done with the distinguishing chromatic number of the complements of $G = C_n(1,k)$, where G is a triangle-free graph. It remains therefore to investigate the distinguishing chromatic number of G, so we can characterize the NGD-graphs. For $G = C_n$ see Theorem 3.5.2. For $G = C_{2k}(1, k-1)$, $k \geq 5$, see Theorem 3.5.1. Moreover, Barrus et al. conjectured in [2] that for $n \geq 10$, $\chi_D(C_n(1,k)) \leq 4$ with few exceptions, unless $G = C_{10}(1,3)$, a bipartite graph of girth 4, in which case the distinguishing chromatic number is shown to be 5 in [2]. Indeed, for all other triangle-free graphs G that are not mentioned in Theorems 3.5.1 and 3.5.2 and $G \neq C_6(1,3)$, $C_8(1,3)$, we shall provide proper distinguishing labelings which show that $\chi_D(G) \leq 5$.

Proposition 3.5.5 *Let* $G = C_n(1,3)$, $n \geq 10$.

- *If* n *is odd, then* $\chi_D(G) = 3$.
- *If* n *is even and greater than 10, then* $\chi_D(G) \leq 4$.
- *If* $n = 10$, *then* G *is a bipartite graph of girth 4 and* $\chi_D(G) \leq 5$.

Proof. We construct a proper distinguishing labeling for each case.

When n is odd, assign label 1 to v_1, v_{n-1} and label 2 to v_n. Starting from v_2, alternate labels 2,3 until v_{n-2}. Since n is odd, v_{n-2} is labeled 3. Clearly, the labeling is proper. Moreover, the only two vertices labeled 1, namely v_1, v_{n-1}, are fixed since their neighborhoods are labeled differently. Thus, there is no nontrivial label-preserving rotation as a symmetry. There cannot be any nontrivial reflection either since v_n, and v_{n-2} are labeled differently. Thus, by Proposition 3.2.5, there is no nontrivial label preserving symmetry. Thus, this labeling is proper and distinguishing. Therefore, we have the desired result.

When n is even, assign label 1, 4 to v_1, v_2 respectively and start from v_3, alternate labels 2,3. Since n is even, v_n is labeled 3. Clearly,the labeling is proper. Moreover, there is no nontrivial label-preserving rotation or reflection as a symmetry since v_1, v_2 are fixed. Thus, by Proposition 3.2.5, there is no nontrivial label preserving symmetry. Thus, this labeling is proper and distinguishing. Therefore, we have the desired result.

When $n = 10$, refer to the labeling in Figure 18. Since the edges are type-1 and type-3, an odd-index vertex can only be adjacent to even-index vertices. Thus, G is bipartite with the odd-index vertices in one partition and the even-index in the other. Observe that any four consecutive vertices form a C_4. Inspection of Figure 18 also reveals that the label is in fact proper and distinguishing. First, realize that the only two vertices labeled 1, namely v_1, v_6 are distinguishable by having distinct neighborhoods; in particular v_1 is adjacent to both vertices labeled 5 whereas v_6 is not. Similarly, both vertices labeled 3, namely v_5, v_7 are distinguishable since v_7 is adjacent to both vertices labeled 5 whereas v_5 has only one neighbor with this label. Similar argument show the vertices labeled 2, 5, 4 are distinguishable. Thus, $\chi_D(C_{10}(1,3)) \leq 5$.

Hence, we have the desired result.

□

We might as well go ahead and characterize the NGD-graphs of the form $G = C_n(1,3)$.

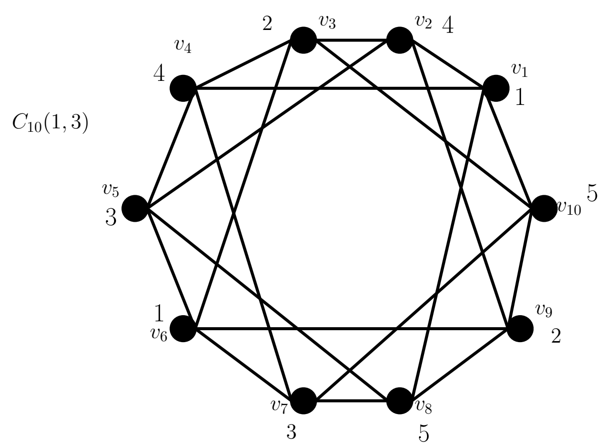

Figure 18. A proper distinguishing labeling of $C_{10}(1,3)$ using 5 labels

Theorem 3.5.6 *Let $G = C_n(1,3)$ be a triangle-free graph. Then G is not a NGD-graph if and only if $n \geq 10$.*

Proof. *Necessity:* By Lemma 3.2.9 and Propositions 3.5.4 and 3.5.5, the result holds for all $n \geq 10$.

Sufficiency: First, note that $C_7(1,3), C_9(1,3)$ are not triangle-free. Thus, due to our main assumption $k \leq \lfloor \frac{n}{2} \rfloor$, it remains to show that the complete bipartite graphs $C_6(1,3), C_8(1,3)$ are both NGD. By Remark 3.2.8 and Proposition 5 in [4], $D(G) = D(\overline{G}) = \chi_D(\overline{G}) = \frac{n}{2} + 1$. Thus, it suffices to show that $\chi_D(G) = n$. Since they are complete bipartite graphs, every pair of nonadjacent vertices have the same neighborhood. Thus, any nonadjacent pair must be labeled differently to obtain a proper distinguishing labeling (see Figure 19). Thus, they are NGD-graphs. Hence, we have the main result.

□

We now characterize the NGD-graphs of the form C_n and $C_{2k}(1, k-1)$, $k \geq 5$.

Theorem 3.5.7 *Let $G = C_{2k}(1, k-1)$ be a triangle-free graph. Then G is not a NGD-graph if and only if $k \geq 5$.*

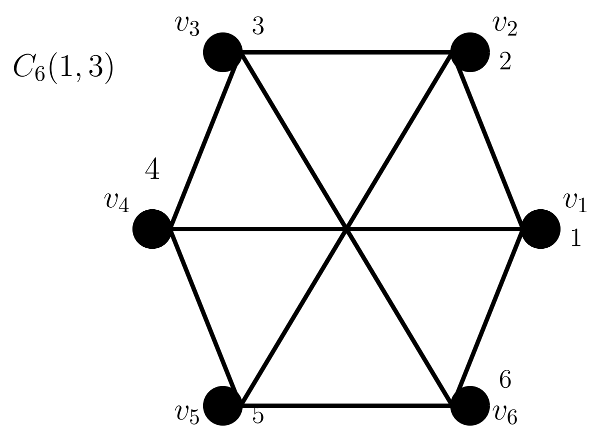

Figure 19. A proper distinguishing labeling of $C_6(1,3)$ using 6 labels

Proof. Necessity follows from Propositions 3.5.4, and 3.5.1 and the fact that $D(G) > 1$.

Sufficiency follows from our main assumption $k \leq \lfloor \frac{n}{2} \rfloor$ and the fact that $C_8(1,3)$ is a NGD-graph as shown in the previous theorem.. □

Theorem 3.5.8 *Let $G = C_n$. Then G is not a NGD-graph if and only if $n \geq 5$.*

Proof. Necessity holds by Theorem 3.5.2 and Proposition 3.5.4. For sufficiency, it suffices to show that for $n = 3$, and 5 , C_n is a NGD-graph. For $n = 3$, result follows by Proposition 3.2.6. Since C_5 is self-complementary, result follows from Theorem 3.5.2.

□

Lastly, we characterize the NGD-graphs for the remaining triangle-free circulant graphs G of the form $C_n(1,k)$.

Proposition 3.5.9 *Let $G = C_n(1,k)$, where $n \geq 8$ and let G be a triangle-free circulant graph such that G is not a cycle graph and $G \neq C_n(1,3)$ and $G \neq C_{2k}(1,k-1)$. Then $\chi_D(G) \leq 5$.*

Proof. Let $n \geq 8$ and $G = C_n(1,k)$ be one of the graphs assumed above. We consider the following two cases:

Case 1. First, let $G = C_{2k}(1, k)$: This is a 3-regular graph. To enhance visualization, consider the following $2 \times k$ table where $B_1 :=$ row 1, $B_2 :=$row 2, where row i represents an ordered arrangement of the set $S = \{3, 4, 5\}$ of labels.

3	4	5	4	5	$\cdots$	5	3
4	5	3	5	3	$\cdots$	3	4

We now proceed with this algorithm:

(1)Assign label 1, 2 to v_1, v_2 respectively.

(2) From v_3 to v_n alternate B_1, B_2 starting with B_1; that is to assign the ordered arrangement of labels in row 1 to $v_3, \cdots, v_{2+k}$ consecutively; then starting at v_{3+k}, do the same with row 2 and so on. Note that the last two labels in row 2 are not assigned.

Clearly from the table and by inspection, we see that for arbitrary j, such that $1 \leq j \leq n$, each of the following pairs of vertices $\{v_j, v_{j+1}\}, \{v_j, v_{j-1}\}, \{v_j, v_{j+k}\}$ have two distinct labels. Since j is arbitrary, we conclude the labeling is a proper one. Next we show that it is distinguishing: clearly, v_1, v_2 are fixed; thus edge v_1v_2 is fixed. Also, observe that for arbitrary j, the set vertices $\{v_j, v_{j+1}, v_{j+k}, v_{j+k+1}\}$ induces a C_4. The vertex v_n does not belong to a 4-cycle with v_2. Since v_n is the only neighbor of v_1 that does not belong to a 4-cycle with v_2, thus v_n is distinguished. This leads to edge v_1v_n being fixed. For similar reasons, v_{n-1} is the only neighbor of v_n that does not belong to a four-cycle with v_1, thus v_{n-1} is distinguished. By similar reasoning, all other vertices are distinguished. Therefore, we have a proper distinguishing labeling as shown in Figure 20.

Case 2. Consider the tetravalent graphs $G = C_n(1, k)$ such that $G \neq C_{2k}(1, k-1)$, and $G \neq C_n(1, 3)$.

To enhance visualization, consider the following $3 \times k$ table where $B_1 :=$ row 1, $B_2 :=$ row 2, $B_3 :=$row 3, where row i represents an arbitrary ordered arrangement of the set $S = \{1, 2, 3, 4, 5\}$ of labels.

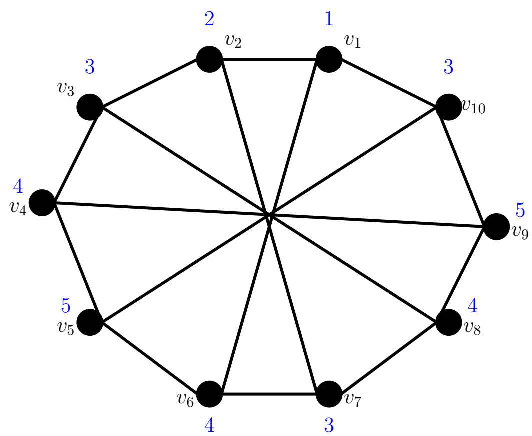

Figure 20. A proper distinguishing labeling of $C_{10}(1,5)$ using 5 labels

B_1	3	2	3	2	3	$\cdots$	3	2
B_2	1	4	5	4	5	$\cdots$	5	1
B_3	4	5	1	5	1	$\cdots$	1	4

We now proceed with this algorithm:

(1) Assign 1 to v_n.

(2) Starting at v_k, assign B_1 to $v_k, \cdots, v_1$.

(3) Starting at v_{n-1}, assign B_1 to $v_{n-1}, \cdots, v_{n-k}$.

(4) From v_{k+1} to $v_{n-(k+1)}$ alternate B_2, B_3.

From the table, since there are no repeating entries in each column, it is clear that the labeling is proper with respect to the subset of vertices $\{v_{k+1}, \cdots, v_{n-(k+1)}\}$. Moreover, a closer examination shall show that it is also proper on the remaining vertices: Realize that going clockwise starting from v_k to $v_{(n-k)}$, two B_1's are used successively with one label 1 used at v_n to offset any chance of potential adjacent vertices having the same label. Table below shows the effect of inserting a label 1 in between 2 B_1's.

3	2	3	2	3	$\cdots$	3	2	
1	3	2	3	2	3	$\cdots$	3	2

Lastly, v_n is not adjacent to any vertex labeled 1. Therefore, the labeling is proper.

It remains to show it is distinguishing. Clearly v_n is distinguished since no other vertex labeled 1 has matching neighborhood. To show that the labeling is distinguishing, we consider these two subcases motivated by the graph automorphism group:

- Let $G = C_n(1,k)$, where $k^2 \equiv \pm 1 \bmod n)$. Observe that for any fixed j, the following sets of vertices $\{v_j, v_{j+1}, v_{j-k}, v_{j-k+1}\}$, $\{v_j, v_{j-1}, v_{j+k}, v_{j+k-1}\}$ form induced 4-cycles. Clearly v_n is fixed (thus distinguished) since no other vertex labeled 1 has a neighborhood receiving the same collection of labels. From our observation, v_{-1} is the only neighbor of v_n that does not belong to a 4-cycle with v_1. Thus, v_{-1} is also distinguished. By the same argument, v_{-2} is also distinguished, which leads to v_{-3} being distinguished also and so on. Eventually, we have all vertices being distinguished (see Figure 21). Hence we have the desired result.

- Let $G = C_n(1,k)$, $k^2 \not\equiv \pm 1 \pmod n)$. Thus by Proposition 3.2.5, $\mathrm{Aut}(G) = D_{2n}$. Since v_n is fixed, thus no nontrivial label-preserving rotation as a symmetry is possible. Moreover, since v_1 is distinguished and v_{n-1}, v_1 have different labels, thus no nontrivial reflection as a symmetry is possible. Therefore, the labeling is distinguishing.

□

Theorem 3.5.10 *Let $G = C_n(1,k)$, be a triangle free graph such that G is neither a cycle graph nor G is one of the following: $C_n(1,3)$ and $C_{2k}(1,k-1)$. Then G is not a NGD graph if and only if $n \geq 5$.*

Proof. Assume that $G = C_n(1,k)$ is a triangle-free graph and $G \neq C_n, C_n(1,3), C_{2k}(1,k-1)$. Necessity follows from Propositions 3.5.4, 3.5.9 and the fact $D(G) > 1$. Sufficiency follows from the assumption $k \leq \lfloor \frac{n}{2} \rfloor$.

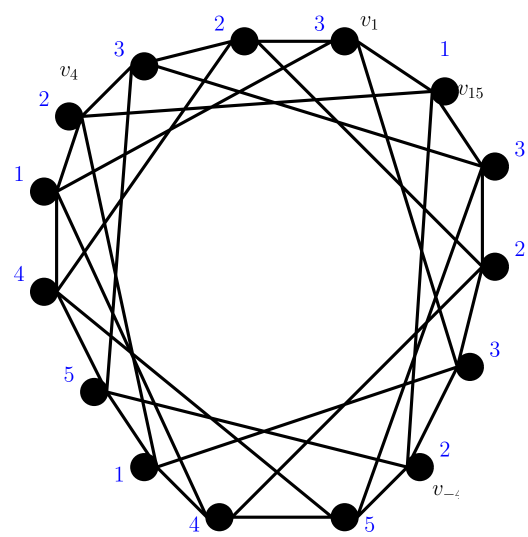

Figure 21. A proper distinguishing labeling of $C_{15}(1,4)$ using 5 labels

□

We are now ready to provide new upper bounds on the sum and the product of the distinguishing chromatic number of a triangle-free graph $G = C_n(1,k)$ and its complement.

Theorem 3.5.11 *Let $G = C_n(1,k)$ be a triangle-free graph and $G \neq C_6(1,3), C_8(1,3)$. Then,*

$$\begin{aligned} 2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) \leq 5 \cdot \left\lceil \frac{n}{2} \right\rceil \\ n \leq \chi_D(G) \cdot \chi_D(\overline{G}) \leq 5 \cdot \left\lceil \frac{n}{2} \right\rceil \end{aligned} \tag{7}$$

Proof.

By remark 3.2.4, the two lower bounds which are the ones stated in [4] remain unchanged. The upper bounds follow from all propositions and theorems in this section. □

Note that $\chi_D(G) \cdot \chi_D(\overline{G}) = \Theta(n)$ in theorem above, improving the order of magnitude from Collins and Trenk.

3.6 Triangle-free circulant graphs $G = C_n(1,S)$, where $S \subseteq \{1, \cdots, \lfloor \frac{n}{2} \rfloor\}$ and the distinguishing chromatic number of their complements

In this section, we extend the result of Theorem 3.2.1 to a larger class of circulant graphs of the form $G = C_n(1,S)$, where $S \subseteq \{1, \cdots, \lfloor \frac{n}{2} \rfloor\}$ and calculate the distinguishing chromatic number of their complements. Without further ado, we state the extended version of Theorem 3.2.1.

Theorem 3.6.1 *Let $S \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, where $n \in \mathbb{N}$ and $n \geq 3$. Then $G = C_n(1,S)$ is a triangle-free circulant graph unless there exists $k, p, q \in S$, where k, p, q are not necessarily different, such that $k + p + q = 0$ mod n, or $k + q = p$.*

Proof. Let $S \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, where $n \in \mathbb{N}$ and $n \geq 3$. Consider $G = C_n(1,S)$. We shall show that if for all $k, p, q \in S$, we have $k + p + q \neq 0$ and $k + p \neq q$, then G is triangle free.

For the sake of contradiction, assume that for all $k, p, q \in S$, we have $k+p+q \neq 0$ and $k+p \neq q$ but G has an induced triangle on the set of vertices v_i, v_j, v_z, where all indices are in mod n. Since v_i, v_j, v_z form an induced triangle, thus $v_i \neq v_j \neq v_z$. Without loss of generality, assume $i < j < z$. Thus, by definition of circulant graph, there exist $k, p, q \in S$ such that $i + k = j$ mod n, $j + p = z$ mod n, and $z + q = i$ mod n. Thus, we have these two possibilities: either $k + p \leq \lfloor \frac{n}{2} \rfloor$ or not. If $k + p \leq \lfloor \frac{n}{2} \rfloor$, then q must be equal to $k + p$ in order to have a triangle. If $k + p > \lfloor \frac{n}{2} \rfloor$, then$k + p + q = 0$ mod n in order to have a triangle. For both possibilities, we have a contradiction. Hence, we have the desired result.

□

Theorem 3.6.2 *Let G be a triangle-free graph of the form $C_n(1,S)$, where $S \subseteq \{1, \cdots, \lfloor \frac{n}{2} \rfloor\}$ then $\chi_D(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof.

Result follows from the proofs of Proposition 3.5.4.

□

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